MATH 417 12/11/2015

④ Symmetric matrices - principal axes.

Example: Find the principal axes of

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix}.$$

Solution:

$$\det(xI - A) = \det\begin{pmatrix} x-4 & -2 \\ -2 & x-7 \end{pmatrix} =$$

$$= (x-4)(x-7) - 4 = x^2 - 11x + 24 =$$

$$= (x-3)(x-8).$$

$\lambda = 3$

$$(3I - A) = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

normalize: $\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$

$\lambda = 8 \qquad (8I - A) = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

normalise: $\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$

$$A \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}$$

orthogonal matrix Q

$$Q^{-1} = Q^T$$

Find an orthogonal matrix Q such that

$$Q^T A Q \text{ is diagonal} \qquad Q_1 = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

(only by chance in this example $Q^T = Q$).

in general, that would not be the case.

Another flavor of the question is: center a conic

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix} \qquad 4x^2 + 4xy + 7y^2$$

1/2 factor

1/2 factor

Center the conic

$$4x^2 + 4xy + 7y^2 = 1$$

and classify it (ellipse or hyperbola)

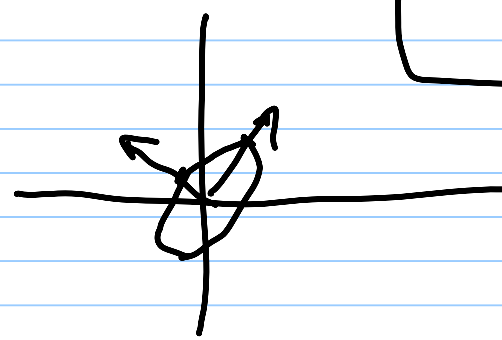
$$Q^T A Q = \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}$$

ellipse

$\boxed{RHS > 0}$

$3 > 0$, $8 > 0$	<u>ellipse</u>
one > 0	one < 0 <u>hyperbola</u>
< 0, < 0	$\emptyset$

$$\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$



(5) Find the principal axes of

$$A = \begin{pmatrix} -2 & 1 & 2 \\ 1 & -2 & 2 \\ 2 & 2 & 1 \end{pmatrix}. \qquad (xI - A) = \begin{pmatrix} x+2 & -1 & -2 \\ -1 & x+2 & -2 \\ -2 & -2 & x-1 \end{pmatrix}$$

$$\det(xI - A) = (x+2)(x+2)(x-1) - 8$$

$$- 4(x+2) - (x-1) - 4(x+2)$$

$= x^3 + 3x^2 - 12 - 9x - 15 = x^3 + 3x^2 - 9x - 27$ $\quad x = -3$

$-27 + 27 + 27 - 27$

$$\begin{array}{r l} & x^2 \qquad - 9 \\ x+3 \,\big) & \overline{x^3 + 3x^2 - 9x - 27} \\ & -x^3 - 3x^2 \end{array}$$

$x = -3$ $\quad -3$ double

$x = +3$

$$\begin{pmatrix} -2 & 1 & 2 \\ 1 & -2 & 2 \\ 2 & 2 & 1 \end{pmatrix} = A. \qquad (-3I - A) = \begin{pmatrix} -1 & -1 & -2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$u = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = v \leftarrow \text{orthogonalise.}$$

$$u \cdot v = 2$$
$$u \cdot u = 2$$
$$v \cdot v = 5$$

keep u , replace v by $v - u\,\frac{u \cdot v}{u \cdot u}$

$$\sim (u \cdot u)v - (u \cdot v)u$$

$$= 2v - 2u = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} \sim \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \xrightarrow{\text{normalise}} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$\lambda = 3$$

$$A = \begin{pmatrix} -2 & 1 & 2 \\ 1 & -2 & 2 \\ 2 & 2 & 1 \end{pmatrix}. \qquad \lambda I - A \quad \begin{pmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & -5 & 2 \\ -2 & -2 & 2 \\ 5 & -1 & -2 \end{pmatrix} \begin{matrix} \\ 2 \\ -5 \end{matrix} \sim \begin{pmatrix} 1 & -5 & 2 \\ 0 & -12 & 6 \\ 0 & 24 & -12 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

Normalise:

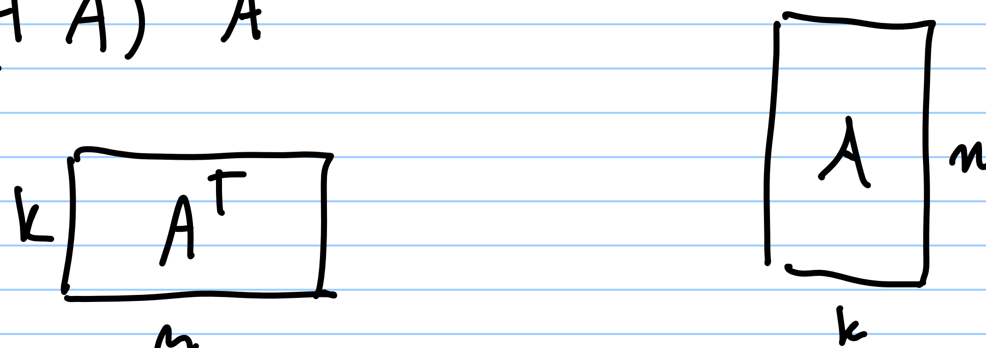
$$\begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \qquad \begin{pmatrix} +1 \\ +1 \\ 2 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 0 \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix}}_{Q^T = Q^{-1}} \begin{pmatrix} -2 & 1 & 2 \\ 1 & -2 & 2 \\ 2 & 2 & 1 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{pmatrix}}_{Q} = \begin{pmatrix} -3 & & \\ & -3 & \\ & & 3 \end{pmatrix}$$

6 orthogonal Projection formula.

Orthogonal Projection matrix onto the space spanned by columns of a matrix A (columns linearly independent!) in $\mathbb{R}^n$

$$A\,(A^TA)^{-1}A^T$$



Example: Find the orthogonal projection matrix from $\mathbb{R}^4$ to the subspace spanned by $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 5 \end{pmatrix}$.

A

$$A^T A = (31)$$

calc full matrix out:

$$A A^T = \begin{pmatrix} 1 & 1 & 2 & 5 \\ 1 & 1 & 2 & 5 \\ 2 & 2 & 4 & 10 \\ 5 & 5 & 10 & 25 \end{pmatrix}$$

Answer:
$$\begin{pmatrix} 1/31 & 1/31 & 2/31 & 5/31 \\ 1/31 & 1/31 & 2/31 & 5/31 \\ 2/31 & 2/31 & 4/31 & 10/31 \\ 5/31 & 5/31 & 10/31 & 25/31 \end{pmatrix}$$

<u>Example 2</u>: Project the vector $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ orthogonally in $\mathbb{R}^3$ onto $\left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 2 \end{pmatrix} \qquad A^T A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 2 \end{pmatrix} =$$

Gramm matrix

$$= \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}$$

inverse: $\frac{1}{14} \begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix}$

$$\frac{1}{14} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 6 & 4 \\ 5 & 1 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} =$$

$$= \frac{1}{14}\begin{pmatrix} 10 & 6 & 2 \\ 6 & 5 & -3 \\ 2 & -3 & 13 \end{pmatrix}$$

$$\frac{1}{14}\begin{pmatrix} 10 & 6 & 2 \\ 6 & 5 & -3 \\ 2 & -3 & 13 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 20 \\ 5 \\ 25 \end{pmatrix}$$

7 Singular values are not eigenvalues,
they are square roots of the eigenvalues
of the Gramm matrix.

The top singular value of A is the maximum
length of a vector Av where $\|v\| = 1$.

<u>Example</u>: Find the singular values of

$$A = \begin{pmatrix} -2 & 2 \\ 3 & 3 \end{pmatrix}.$$

$$A^T A = \begin{pmatrix} -2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 5 \\ 5 & 13 \end{pmatrix}$$

$$\det(xI - A^T A) = \det \begin{pmatrix} x-13 & -5 \\ -5 & x-13 \end{pmatrix} = x^2 - 26x + 169 - 25$$

$$= x^2 - 26x + 144$$

$=$

$$\lambda = \frac{26 \pm \sqrt{26^2 - 144 \cdot 4}}{2} = 13 \pm \sqrt{169 - 144} =$$

$$= 13 \pm 5 \quad \begin{cases} 18 \\ 8 \end{cases}$$

Answer to problem: $\sqrt{18}$, $\sqrt{8}$

singular values of A.