Covering Space Theory

$X$ connected CW-complex, $x \in X$

Covering:

$Y$  

$X$  

fundamental neighborhoods

locally (over some open set $U_i \subseteq X$ which cover $X$),

a product with a discrete set

Based indexings:

* $y \in Y$ choose a base point $y^*$

$X$  

$Y$  

homeomorphism of based coverings.
\[ \pi_1(f) : \pi_1(Y) \rightarrow \pi_1(X) \] for a based covers

inj

and in fact defines an equivalence of categories

\( (\text{connected based covers, morph.}) \rightarrow (\text{subgroups of } \pi_1(X), \text{ inclusions}) \)

Choose a different base point of \( i \) over \( x \):

\( \text{corresponds to conjugate subgroup} \)

If all these are isomorphic in the based category \( i \) is called

(choosing different base points \( x \))

\( \text{a regular covering} \)

\( \text{This corresponds to normal subgroups of } \pi_1(X). \)
Example: Is a regular covering of a regular covering a regular covering?

No

\[ H \leftarrow K \leftarrow G \]

\[ H \leftarrow G \]

B/H acts by permuting factors.

Normal subgroups, not every subgroup is normal under the multiplication of factors.
All maps (unbased) $y \rightarrow y'$ morphisms (unbased covering base category)

$f^{-1}(+) \to \pi_1(X,+) - \text{action}$

This defines an equivalence of categories between all unbased $X$-covering spaces and $\pi_1(X,+)$-sets.
What does the category of $G$-sets look like?

$G$-sets $S = \prod G/H_i$

\[ \text{Full subcategory of } G\text{-sets} \]

$G/H \rightarrow G/K$

\[ 1 \rightarrow g \]

\[ h \rightarrow h g = g k \]

$g \in G/K, g^{-1} H g \leq K$

$\theta_c (G/H, G/K)$

Sub-conjugacy class
Example: let $\pi_1(X_p) = \mathbb{Z}_p$. How many non-homeomorphic $X \times \mathbb{C}^\ast$ are there?

(a) based connected ones
(b) unbased connected ones

are there?

- subgroups of $\mathbb{Z}_p$

$\mathbb{Z}_p = \langle (1,2) \rangle \langle (1,3) \rangle \langle (2,3) \rangle \langle A_3 \rangle$

- Conjugate subgroups $\cong$?

Answer: $4$
Graphs and groups

A covering space of a graph is a graph.
A subgroup of a free group is free.

Example:

Find free generators $\pi_1(Y, x) \cap <a, b>$.

4 generators:
- $aba^{-1}$
- $a$
- $a^{-1}ba$
- $b$
$N \triangleleft F \rightarrow F/N$

$N$ is free. What are its free generators?

If $F = \Pi_1(X)$ where $X_0 = \ast$, CW complex $X_1 = X$

Then the covering is the Cayley graph of $F/N$ corresponding to $N$ with respect to the free generators of $F$. 
Most general case: \( H \subset F \rightarrow F/H \) in general not a group

\[ H = \pi_1(S+, \text{Schreier graph}) \]

(generator of a group acting on \( \pi_0 \) coset)

Example: Find free generators of the subgroup

\( \langle a, b \rangle \) generated by

1. all conjugates of \( a^2, b^2, ba^2a^{-1} \)
2. all the elements in \( a \) and \( b \).
\langle a, b \mid a^3, b^2, aba^{-1}b^{-1} \rangle 

\text{3-cycle 1-2-cyle} = \Sigma_3

\text{not a regular cover}
Homology of a CW-complex

\[ X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \]

Chain complex

\[ \mathbb{Z}[I_0] \supseteq \mathbb{Z}[I_1] \supseteq \ldots \supseteq \mathbb{Z}[I_n] \]

free abelian group

\[ dd = 0 \]

\[ H_r(X) = \text{Ker } d / \text{Im } d \]

Example: \( H_k \mathbb{C}P^n = \mathbb{Z} \) \( \text{if } k \text{ even } \leq 2n \)

\[ Z \subseteq 0 \subseteq Z \subseteq 0 \subseteq \mathbb{Z} \subseteq \ldots \]

\[ d : \mathbb{Z}[I_n] \rightarrow \mathbb{Z}[I_{n+1}] \]

\[ (I_{n+1}) \times |I_n| \text{ matrix of integers} \]
What is the $(i,j)$ entry?

Incidences number

\[
\begin{array}{ccc}
\text{m-cell} & \rightarrow & \text{degree} \\
\text{degree} & \rightarrow & \text{map} \\
S^m & \rightarrow & X_{m-1} & \rightarrow & S^{m-1}
\end{array}
\]

\[\text{degree, } \deg f = \sum \text{sign} (\det Df_y) \quad \text{for } y \in f^{-1}(x)\]

For $n = 1$:
- beginning point
- end point

\[\text{everythings except the i-cell.}\]
The degree makes sense \( \text{mod } 2 \) even when \( M \) is not orientable. \( \mathbb{R}P^2 \to S^2 \) 

Example: \( \mathbb{R}P^n \)

\[
\begin{array}{cccccc}
0 & 2 & 0 & \cdots & 0 & 2 \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \cdots & \mathbb{Z} & \mathbb{Z}
\end{array}
\]

\( H_k(\mathbb{R}P^n) = \mathbb{Z} \) for:
- \( k = 0 \)
- \( k = m \) odd
- \( 0 < k < m \) odd

Orientation:
- Same if \( n-1 \) even
- Opposite if \( n-1 \) odd
Example:

\[ \mathbb{R}^n \rightarrow \mathbb{R}^n \]

Let \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a function defined by \( f(x) = x \). For every \( x \in \mathbb{R}^n \), the function maps \( x \) to itself.

Consider \( \mathbb{R}^n \) as a Euclidean space, which has a natural inner product structure.

Let \( x \in \mathbb{R}^n \). Then, \( x \cdot x = \|x\|^2 \), where \( \|x\| \) is the norm of \( x \).

For any \( x \), \( x \cdot x \geq 0 \), and equality holds if and only if \( x = 0 \).
Euler characteristic

\[ \chi(X) = \sum_k (-1)^k \text{rank } H_k(X) \]

finite CW-complex

Lefschetz Theorem

\[ \chi(f) = \sum_k (-1)^k \text{Tr } H_k(f) \]

\[ \chi(f) \neq 0 \Rightarrow f \text{ has a fixed point} \]
Local homology

How to tell a space $X$ is not a topological manifold? (w/o boundary)

If $X$ is a manifold:

- locally $\cong \mathbb{R}^n$
- $H_k(X, X \setminus \{x\}) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{else} \end{cases}$

Example: $X = (\mathbb{R}^n / x \sim -x)$ For what $n$ is this a manifold?

- $n = 1$ \quad no
- $n = 2$ \quad yes $\cong \mathbb{R}^2$
$m \geq 2$, $x = 0$:

\[ H_n(X_1, X \setminus \text{los}) \cong \tilde{H}_{k-1}(\mathbb{R}P^{n-1}) \]

\[ \text{intertwined} \quad (0, \infty) \times \mathbb{R}P^{n-1} \quad \text{non-zero for } m > 1 \text{ dimension} \]

NO for $m > 2$. 