MATH 286 HANDOUT 3: SOME BASIC THEORY FOR SYSTEMS OF ODE'S

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First of all, what is a general system of ordinary differential equations? There are several unknown functions $y_1, ..., y_n$ of the same independent variable t. We are given n equations relating t and values of the y_i's, and their higher derivatives $y_i^{(j)}$, at the point t. For each i, a particular highest derivative $y_i^{(k_i)}$ for some number k_i will occur in the system. We assume this highest derivative is *explicit* in the system, i.e. occupies the entire left hand side of an equation, and occurs nowhere else. Therefore, the system looks as follows:

$$
\begin{aligned}
&y_1^{(k_1)} = f_1(t, y_1, y_1', ..., y_1^{(k_1-1)}, ..., y_n, ..., y_n^{(k_n-1)})\\
&y_2^{(k_2)} = f_2(t, y_1, y_1', ..., y_1^{(k_1-1)}, ..., y_n, ..., y_n^{(k_n-1)})\\
&...\\
&y_n^{(k_n)} = f_n(t, y_1, y_1', ..., y_1^{(k_1-1)}, ..., y_n, ..., y_n^{(k_n-1)}).
\end{aligned}
\tag{1}
$$

We see that the known functions $f_1, ..., f_n$ on the right hand side depend on many variables: t, and all the derivatives of the functions y_i up to the $k_i - 1$'st derivative, i.e. the derivative *just below* the highest derivative of y_i which occurs in the system (if the highest derivative also occured, it would not have been *explicit* in our sense).

What a mess! It is difficult to do anything with a system written in form as complicated as (1). A much nicer object of study is a *first order system of ODE's*, in which only the first derivatives of the functions $y_1, ...y_n$ occur, and only explicitly, on the left hand sides of the equations:

$$
\begin{aligned}
&y_1' = f_1(t, y_1, ..., y_n)\\
&y_2' = f_2(t, y_1, ..., y_n)\\
&...\\
&y_n' = f_n(t, y_1, ..., y_n).
\end{aligned}
\tag{2}
$$

Here, $f_1, ...f_n$ are known functions of "just" $n+1$ variables.

But isn't (2) too much of a restriction? Surely, higher derivatives will occur in practical problems. But it turns out that there is a very simple and general way of always *converting* a system of the form (1) to a system of the form (2), although at the price of *increasing* the "size" of the system, i.e. the number of unknown functions. The idea is to introduce "auxilliary" unknown functions y_{ij}, which are the derivatives, and higher derivatives, of the functions y_i, up to $k_i - 1$'st derivative, i.e. "just below" the highest derivative, which is k_i'th. This means

$$y_{ij} = y_i^{(j)},\ j = 0, ..., k_i - 1.$$

Then the system (1) can be written as a first order system

$$(3)\qquad \begin{aligned}
&y'_{1,k_1-1} = f_1(t, y_{1,0}, ..., y_{1,k_1-1}, ..., y_{n,0}, ..., y_{n,k_n-1}),\\
&y'_{2,k_2-1} = f_2(t, y_{1,0}, ..., y_{1,k_1-1}, ..., y_{n,0}, ..., y_{n,k_n-1}),\\
&...\\
&y'_{n,k_n-1} = f_n(t, y_{1,0}, ..., y_{1,k_1-1}, ..., y_{n,0}, ..., y_{n,k_n-1}),\\
&y'_{1,0} = y_{1,1},\\
&...\\
&y'_{1,k_1-2} = y_{1,k_1-1},\\
&...\\
&y'_{n,0} = y_{n,1},\\
&...\\
&y'_{n,k_n-2} = y_{n,k_n-1}.
\end{aligned}$$

Make sure you don't forget the "trivial" equations in (3), relating $y_{i,j}$ and $y_{i,j+1}$!

A case of the system (2) which deserves special attention is a *linear* system

$$(4)\qquad \begin{aligned}
&y'_1 = g_{11}y_1 + ... + g_{1n}y_n + r_1\\
&...\\
&y'_n = g_{n1}y_1 + ... + g_{nn}y_n + r_n.
\end{aligned}$$

Here g_{ij}, r_i, $i = 1, ..., n$ are known *functions* of the one variable t. (4) is a *first order* linear system, but we can similarly define higher order linear systems, which can then always be converted to first order linear systems using the formula (3). This is in particular important in the case of a single n-order differential equation, which we can thus convert to a first order linear system of n differential equations. We will use that later in the class!

Example 1: Convert to a first order system of ODE's:

$$\begin{aligned}
x'' &= \ln(x + t^2 + y'') + e^{x't}\\
y''' &= \sin(y'' + x't^2).
\end{aligned}$$

Answer: Putting $u = x', v = y', w = y''$, the system can be converted to

$$\begin{aligned}
u' &= \ln(x + t^2 + w) + e^{ut}\\
w' &= \sin(w + ut^2)\\
x' &= u\\
y' &= v\\
v' &= w.
\end{aligned}$$

Example 2: Convert the LDE

$$(5)\qquad y'' = e^t y' + t^2 y + \ln(t)$$

into a system of first order LDE's. Answer: Put $x = y'$, the system will be

$$(6)\qquad \begin{aligned}
x' &= e^t x + t^2 y + \ln(t)\\
y' &= x.
\end{aligned}$$

It is important to know that systems of first order LDE's obey existence and uniqueness theorems completely analogous to those valid for single first order ODE's:

Theorem 1. *Suppose $n+1$-variable functions $f_1(t, y_1, ..., y_n), ..., f_n(t, y_1, ..., y_n)$ are defined and continuous for $a < t < b$, $c_i < y_i < d_i$ and suppose that in those bounds, $\frac{\partial f_i}{\partial y_j}$ are also defined and continuous. Suppose further $t_0, y_{10}, ..., y_{n0}$ are numbers such that $a < t_0 < b$, $c_i < y_{i0} < d_i$. Then there exists a number $\epsilon > 0$ such that the initial value problem (2) with initial conditions*

$$y_i(t_0) = y_{i0}, \quad i = 1, ..., n$$

has a unique solution defined on the interval $t_0 - \epsilon < t < t_0 + \epsilon$.

As usual, there is a stronger theorem for linear systems:

Theorem 2. *Suppose functions $g_{ij}(t)$, $r_i(t)$ are defined and continuous for $a < t < b$. Suppose further $t_0, y_{10}, ..., y_{n0}$ are numbers such that $a < t_0 < b$. Then the initial value problem (4) with initial condition*

$$y_i(t_0) = y_{i0}, \quad i = 1, ..., n$$

has a unique solution defined on the interval (a, b).

Example 3: Find the domain of definition of the initial value problem involving the equation (5) of Example 2 with initial conditions

$$y(1) = 20, \ y'(1) = -99.$$

Solution: When we convert the equation (5) to the form (6), our IVP says

$$y(1) = 20, \ x(1) = -99.$$

So Theorem 2 applies. So the answer is $(0, \infty)$.