MATH 395 PROBLEMS 12

IGOR KRIZ

Regular problems:

1. Recall that we say that $\int f$ *exists* if the function f is Lebesgue-integrable, and that $\int f$ *converges* if it exists and is finite. For what values of a do the following integrals converge:

$$\int_0^1 x^a dx, \quad \int_1^\infty x^a dx, \quad \int_0^\infty x^a dx?$$

2. Compute

$$\int_{[0,1]\times[0,1]} \max(x,y) dxdy.$$

[Use Fubini's theorem, but not mechanically.]

3. Decide whether the following integrals exist and/or converge: [use bounds by integrals you know]
(a) $\int_2^\infty \frac{dx}{\sqrt{x^4-1}}$
(b) $\int_0^\pi \frac{\sin(x)}{x} dx$
(c) $\int \frac{dx}{x^2+x}$.

4. Let $f : [0,\infty) \to \mathbb{R}$ be defined by $f(0) = 0$ and

$$f(x) = \frac{(-1)^n}{n} \text{ for } x \in (n-1, n].$$

Prove that the Lebesgue integral $\int f$ does not exist, but the improper Riemann integral

$$\lim_{t\to\infty} \int_0^t f(x)dx$$

exists. Is there such an example with $f \geq 0$? [In proving the non-existence, use the fact that an $g \in Z^{inc}$ such that $g \geq f$ is bounded below by a function from Z, and hence is non-negative outside of a bounded interval; similarly, $g \in Z^{dec}$ such that $g \leq f$ is non-positive outside of a bounded interval; conclude that the upper integral of f is $+\infty$, the lower integral is $-\infty$.]

Challenge problems:

5. Define the ζ-function $\zeta : (1, \infty) \to \mathbb{R}$ by

$$\zeta(x) = \sum_{n=1}^{\infty} n^x.$$

Does the Lebesgue integral

$$\int_1^2 \zeta(x) dx$$

converge?