MATH 395 PROBLEMS 13

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Regular problems:

1. For which values of $t > 0$ does the following integral converge:

$$\int_0^\infty \frac{e^{-x}}{x^t} dx?$$

2. Do the following integrals exist/converge?
(a) $\int_1^\infty \frac{\sin x}{x^2} dx$
(b) $\int_{-\infty}^\infty \frac{1}{x} dx$.
[Recall that if we put $f^+ = \max(f, 0)$ and $f^- = \min(f, 0)$ then $\int f$ exists if and only if both $\int f^+, \int f^-$ exist and at least one converges, and $\int f$ converges if and only if both $\int f^+, \int f^-$ converge.]

3. Referring to your notes, theorems from class, or using your own argument, show that if $\int f$ exists and $f(x) = g(x)$ for all but countably many points x, then $\int g$ ecists and $\int g = \int f$.

4. Find a closed subset of $[0, 1]$ which contains no non-empty open interval, but is not of measure 0. (Recall that a subset of a metric space is closed if its complement is open.) [Hint: use the construction of the Cantor set by omitting open intervals; make arrangements so that the total length of the intervals omitted is < 1.]

Challenge problem:

5. Prove that a bounded function f on a bounded closed interval $[a, b]$ is Riemann-integrable if it is continuous almost everywhere. [Cover the set of points of discontinuity by countably many open intervals with small total length, and the points of continuity by open intervals in which the function does not change values by more that ϵ. You can assume that there are only countably many intervals involved in the covering, say, by arranging that their endpoints be rational... Then finitely many of these intervals will also cover $[a, b]$ (because if there were an x_n in the complement of the first n intervals, then the limit could not be in any of the intervals - a contradiction).]

Double challenge: prove that the "only if" statement is also true.