MATH 395 PROBLEMS 4

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Regular problems:

1. Rewrite the following system of differential equations in the form $y_i' = f_i(x, y_1, \ldots, y_n)$, $i = 1, \ldots, n$:

$$z_1'''' = g_1(x, z_1, z_2, z_1', z_2', z_1'', z_2'', z_1''', z_2'''),$$
$$z_2''' = g_2(x, z_1, z_2, z_1', z_2', z_1'', z_2'', z_1''').$$

2.
(a) Two $n \times n$ matrices A, B are called *similar* if there exists an invertible $n \times n$ matrix P such that $A = P^{-1}BP$. Prove that similar matrices have the same eigenvalues.
(b) Find an explicit formula for the eigenvalues of a 2×2 matrix.

3. A *normed space* is a vector space V over $\mathbb{R}$ together with a function (called norm) assigning to every $x \in V$ a number $||x|| \geq 0$ such that (1) $||x|| = 0$ iff $x = 0$, (2) $||x + y|| \leq ||x|| + ||y||$. Prove that if V is a normed space, then V is a metric space with metric

$$\rho(x, y) = ||x - y||.$$

Give at least three different examples of norms on $\mathbb{R}^n$. (Challenge: can you give infinitely many examples?)

4. An $n \times n$ matrix is called a *projection matrix* if $AA = A$. Prove an eigenvalue of a projection matrix is either 0 or 1.

Challenge problems:

5. Let $\mathbb{R}^n$ be a metric space with metric

$$\rho((x_1, ...x_n), (y_1, ..., y_n)) = \max_{i=1,...,n} |x_i - y_i|$$

and let $f : \mathbb{R} \to \mathbb{R}$ be a map such that for some constant K,

$$\rho(f(x), f(y)) < K\rho(x, y)$$

for every $x, y \in \mathbb{R}$. Prove that there exists a $\lambda \in \mathbb{R}$ and an $x \in \mathbb{R}^n$ such that

$$f(x) = \lambda x.$$

Is this true if the above metric is replaced by the Euclidean metric?

6. Give an example of a metric d on $\mathbb{R}^n$ such that $(\mathbb{R}^n, d)$ is homeomorhpic to $(\mathbb{R}^n, \rho)$ (ρ is defined in Problem 5), but such that $(\mathbb{R}^n, d)$ is not complete.