MATH 396 PROBLEMS 10

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Regular problems:

1. Calculate the integral of the function
$$f(z) = \frac{3z-4}{z^2-3z+2}$$
over
(a) the circle with center 0 and radius 3/2 oriented counter-clockwise
(b) the circle with center 0 and radius 3 oriented clockwise.
[Use the method of partial fractions.]

2. Are the following functions holomorphic?
(a) $f(x+iy) = \cos(x^2+y^2)(e^{2xy}+e^{-2xy}) + i\sin(x^2+y^2)(e^{-2xy}-e^{2xy})$.
(b) $g(x+iy) = e^{(e^x\cos y)}\cos(e^x\sin y) + ie^{(e^x\cos y)}\sin(e^x\sin y)$.

3. A holomorphic function $f : \mathbb{C} - \{0\} \to \mathbb{C}$ has the property that $\int_L f = 0$ for any circle L in $\mathbb{C} - \{0\}$ with radius 1. Is f necessarily constant? (Give a reason.)

4. Calculate (from first principles) the complex derivatives of all the functions z^n, $n \in \mathbb{Z}$.

Challenge problem:

5. Let $p(z)$ be a polynomial with complex coefficients which has no roots of higher multiplicities. Prove that if L is a piecewise smooth Jordan curve oriented counter-clockwise, then the number of roots of p which are located inside L is equal to

$$\frac{1}{2\pi i}\int_L \frac{p'(z)}{p(z)}.$$

[First realize that the integral is 0 around areas in which p has no roots. Thus, it suffices to consider the case when L is a very small circle inside of which there is exactly one root x. In this case, write $p(z) = (z - x)q(z)$ where q has no roots inside L. Rewrite the integral in terms of q, and use the facts we already know.]