MATH 396 PROBLEMS 11

IGOR KRIZ

Regular problems:

1. Prove that the function

$$f(x+iy) = \frac{x+1}{(x+1)^2+y^2} - i\frac{y}{(x+1)^2+y^2}$$

is holomorphic where defined. What is its domain of definition?

2. Using the Caychy formula, calculate:
(a) $\int_K \frac{e^x}{(x-2)^3}dx$ where K is a small circle around 2 oriented counter-clockwise
(b) $\int_L \frac{e^z+z}{z^2(z-1)}dz$ where L is the circle with center 0 and radius 6, oriented counter-clockwise. [For this one, use the method of partial fractions (ignoring the numerator), to get denominators as in the Cauchy formula.]

3. Using the definitions of complex elementary functions, rewrite the following expressions of $z = x+iy$ as $f_1 + if_2$ where f_1, f_2 are real elementary functions of x and y (e.g., $e^z = e^x\cos(y) + ie^x\sin(y)$):
(a) e^{z^2}
(b) $\frac{1}{z^2+e^z}$.

4. Find the Taylor expansion of the function $z\ln(z)$ with center 1. What is its radius of convergence?

Challenge problem:

5. A point $a \in \mathbb{C}$ is called an *isolated singularity* of a function f if f is holomorphic in some disk with center a, but with a removed (i.e. $B_\epsilon(a) \setminus \{a\}$). a is called a *removable singularity* if f is bounded in a neighbourhood of a. a is called a *pole* if $f(z)(z-a)^n$ has a removable singularity at a. Otherwise, a is called an *essential singularity.*

(a) Prove that if a is a removable singularity, then one can redefine $f(a)$ in such a way that f becomes holomorhpic in $B_\epsilon(a)$. [Use the Cauchy formula.]

(b) Prove that if f is holomorphic at a, then $1/f$ is holomorphic or has a pole at a (i.e. does not have an essential singularity). [Use the Taylor formula.]

(c) Prove that the function $e^{-1/z}$ has an essential singularity at 0.

(d) Prove that if a function has an essential singularity at a, then the image $f[B_\epsilon(a)] = \{f(z)|z \in B_\epsilon(a)\}$ is dense in $\mathbb{C}$ (i.e. every non-empty open set in $\mathbb{C}$ intersects $f[B_\epsilon(a)]$). [Hint: use contradiction and (b).]