MATH 396 PROBLEMS 4

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Regular problems:

1. Find the equation (in Cartesian coordiantes) of the ellipse with foci $(c, 0)$, $(-c, 0)$ and main axis a (i.e. the set of all points in the plane whose distances to the two foci add up to $2a$).

2. The moon of a planet moves according the equation

$$r = \frac{1}{2 + cos(\alpha)}$$

in polar coordinates with origin in the planet. What is the area swept by the moon (in the sense of Kepler's law) in a quarter of a month (where the month is defined by that moon)? (Suppress units.) [Hint: if you choose to solve this problem by integrating in polar coordinates, remember the substitution $t = \tan(\alpha/2)$. Alternately, you could use the major and minor axes.]

3. Consider the curve given by the equation $r = \alpha^2$ in polar coordinates. Calculate the area swept by the curve between the angles $\alpha = 0$, $\alpha = \pi/3$.

4. Consider the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(u, v) \mapsto (x, y)$$

where

$$x = u + v, \quad y = v - u^2.$$

Let T be the triangle with vertices $(0, 0)$, $(2, 0)$, $(0, 2)$. Let $S = f[T]$ (the image of T under f). Using substitution, calculate

$$\int_S (x - y + 1)^{-2} dxdy.$$

Challenge problem:

5. Consider the parabola $y = ax^2$ where a is a constant. The *focus* of the parabola is a point F on the y axis such that for any point P on the parabola, the angle between the line segment PF and the tangent at P is equal to the angle between the tangent at P and a vertical (=parallel to the y-axis) line through P.

(a) Find the focus of the parabola, and prove that it is a focus.

(b) Shifting the parabola so that the focus is in the origin, find its equation in polar coordinates.