How do you write down the effect of a move?

Corner pieces

e.q.

Disregarding orientation, how do the pieces move around?
Let us study a finite set, and ways of moving its elements around (permutations):

Example: \{1, 2, 3\}

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{array}
\]

6 permutations on a 3-element set
On an $n$-element set, I have $n!$ permutations.

The group of all permutations on $\{1, \ldots, n\}$ is called the symmetric group $\Sigma_n$. (Some books use $S_n$.)
The order of the group \( \Sigma_n \) (= \# of elements).

Subgroups of \( \Sigma_n \) are called permutation groups.

**Example:** Even and odd permutations.

Recall the determinant:

\[
\det \begin{pmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33}
\end{pmatrix} = a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31}
\]

\( a_{ij} \in \mathbb{R} \)
Even permutations get +

Odd permutations get −

The mental element is even.

A simple switch of just two elements (and nothing else) is odd.

Example: \(1 \ 2 \ 3 \ 4 \ 5\) \(\leftrightarrow\) is this even or odd?

\[5 \ 3 \ 4 \ 1 \ 2\] How many simple switches does it take to get back?
HW: 1 2 3 4 5 6

1 2 4 5 3
1 2 3 5 4
1 2 3 4 5

4 switches = an even permutation.

Is it even or odd?

(odd number of switches would have been odd).

To prove that even or odd is uniquely determined, we need another method for figuring out
If a permutation is even or odd:

Here it is: Look at our examples again!

\[1 \ 2 \ 3 \ 4 \ 5\]
\[5 \ 3 \ 4 \ 1 \ 2\]

How many pairs of numbers got switched?

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8 pairs switched \(\Rightarrow\) even
An even number of pairs switched \(\Rightarrow\) even
An odd number of pairs switched \(\Rightarrow\) odd.

A statement to help our proof.

**Lemma:** If we have calculated the evenness (or oddness) of a permutation using this method and then we perform one switch only, the result (= the parity) will change.

Proof: Imagine we scramble the \(\{1, \ldots, n\}\) set and then perform one switch.

\[ i \xrightarrow{\text{switch}} i' \quad \text{WLOG (without loss)} \]
How does it affect the number of pairs
switched? the switching of the pair \( (i, j) \)
in the \( i \)'th place and \( j \)'th place will change.
Counting I change.

• Whatever pair does not have either \( i \) or \( j \)
will not change (no more).
Before i'th spot with j', no change.

\[ \text{Before} \quad i'th \quad \text{spot} \quad \rightarrow \quad \text{no change} \]

Similarly, whatever is after j'th spot, no change.

\[ \text{i'} \quad \leftarrow \quad j' \quad \rightarrow \quad \text{no change} \]

What about the element between the
Yes!  These will change!!

There is an even number of these!

Pair the third element with $i'$ or $j'$.

Even number will not affect parity, so we are done!
The set of all even permutations or
the set $\{1, \ldots, n\}$ forms a subgroup.

Even permutation

Even permutation

Even # of switches

Even # of switches

Total # of switches.

This is called the alternating group $A_n$. 
Theorem about Rubik's cube: A position on Rubik's cube that switches two pieces only (and nothing else) cannot be solved (or achieved by legal move).

Proof: Turning one face 90° is an even permutation.

HW 2: Prove it.

So any permutation on the cube is even! □