$S = \{1, 2, 3\}$
$T = \{4, 5, 6\}$
$U = \{1, 7, 8\}$

$f : S \rightarrow T$
$g : T \rightarrow U$

$f(1) = 4, f(2) = 4, f(3) = 6$
$g(4) = 7, g(5) = 1, g(6) = 8$

**Composition:**

$\text{Composition:}$
\[ g(f(1)) = g(4) = 7 \]
\[ g(f(2)) = g(4) = 7 \]
\[ g(f(3)) = g(6) = 8 \]

Because of that, the composed mapping is denoted by \( g \circ f \) (Note: in contrast with our notation for permutations.)

\[ g \circ f : S \rightarrow U \]

The identity mapping

\[ \text{Id} = \text{Id}_S : S \rightarrow S \]

\[ \text{Id}(x) = x \]

"the function x"
The inverse of a mapping \( f : S \to T \) (if one exists) is mapping \( f^{-1} : T \to S \) such that \( f \circ f^{-1} = \text{Id}_S \) and \( f^{-1} \circ f = \text{Id}_T \).

Note that a mapping has an inverse if and only if it is bijective (is a bijection).
Recall: A homomorphism of groups is a mapping such that
\[ f(xy) = f(x)f(y) \quad \text{for all} \quad x, y \in G. \]
(\text{It follows that} \quad f(e) = e, \quad f(x^{-1}) = (f(x))^{-1}.)
Note: An isomorphism is the same thing as a bijection homomorphism.

Other examples of homomorphisms: If $f_1 \subseteq G$ is a subgroup,

\[ i : H \rightarrow G \]

\[ i(x) = x \]

\[ \text{injective} \]
\[ \text{is a homomorphism (called an inclusion).} \]

\[ \text{(it is not called the identity because the domain is not equal to the codomain).} \]
Another example is when \( H \triangleleft G \), then we have the factor group

\[
G/H = \{ gH \mid g \in G \} = \{ \{ h \mid g \in G \} \}
\]

\[
(g_1 H)(g_2 H) = g_1 g_2 H.
\]

Then \( p : G \to G/H \) given by \( p(g) = gH \) is a homomorphism (called the projection).

\( |G| = |G/H| \cdot |H| \).

(recall "the equivalence class of \( g \),")
The homomorphism theorem: Let \( f : G \to J \) be any homomorphism of groups. Define the kernel of \( f \) as

\[
\text{Ker } f = \{ g \in G \mid f(g) = e \}
\]

the set of all \( g \in G \) such that \( f(g) = e \).

Then \( \text{Ker } f \triangleleft G \) and there exists a unique

\( p : G \to G/\text{Ker } f \)

such that \( g \mapsto g \text{Ker } f \).
injective homomorphism

\[ \overline{f} : G/\text{Ker}f \rightarrow J \quad \text{(only one)} \]

such that \( f \circ p = \overline{f} \).

Then two mean the same thing.

\[ p : G \rightarrow G/\text{Ker}f \]

projection

\[ G \xrightarrow{f} J \]

Example: \( f : (\mathbb{Z}/8,+) \rightarrow (\mathbb{Z}/8,+) \) (\( \ast \))

\[ f(x) = 2x \mod 8 \]
\[ \mathbb{Z}/8 = \{0, 1, 2, 3, 4, 5, 6, 7\} \]

\[ \text{Ker} f = \{0, 4\} \]

\[ \mathbb{Z}/8 / \{0, 4\} = \{[0], [1], [2], [3]\} \]

\[ \mathbb{Z}/4 = \{[0], [1], [2], [3]\} \]

(\text{Wait a while, I will use the homomorphism theorem to prove this.})

\[ \mathbb{Z}/8 \xrightarrow{\cdot 2} \mathbb{Z}/8 \]

\[ \mathbb{Z}/8 / \{0, 4\} \]

\[ \{0\} = [0] \]

\[ \{2\} = [2] \]

\[ \{4\} = [4] \]

\[ \{6\} = [6] \]
Note: Now we have a more advanced view of $\mathbb{Z}/8$ (or any $\mathbb{Z}/n$).

Let $m \mathbb{Z}$ be the subgroup of $(\mathbb{Z}, +)$ consisting of numbers divisible by $m$. ($\mathbb{Z}$ is abelian, so every subgroup is normal).

$$\mathbb{Z}/m \mathbb{Z} = \{0 + m \mathbb{Z}, 1 + m \mathbb{Z}, \ldots, (m-1) + m \mathbb{Z}\}$$

$$\mathbb{Z}/n \mathbb{Z} = \{0, 1, \ldots, (n-1)\}$$
\[ \mathbb{Z}/8\mathbb{Z} = \{ 0 + 8\mathbb{Z}, 1 + 8\mathbb{Z}, \ldots, 7 + 8\mathbb{Z} \} \]

\[ 1 + 8\mathbb{Z} = 9 + 8\mathbb{Z} = -7 + 8\mathbb{Z} \ldots \]

(wherever I add a multiple of 8, I get the same coset)

Proof that \( f: \mathbb{Z}/8 \rightarrow \mathbb{Z}/8 \)

is a homomorphism:

\[ 2(a + b) = 2a + 2b \]

\[ x \mapsto 2x \quad \text{is a homomorphism} \]

\[ \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/8\mathbb{Z} \]

projection

\[ \mathbb{Z} \rightarrow \mathbb{Z}/8\mathbb{Z} \]
\[ \text{Ker } g = 4\mathbb{Z} \quad \downarrow \quad \cdots \quad \rightarrow \]
\[ \mathbb{Z}/4\mathbb{Z} \]

(I need a slightly more general statement, will continue next time.)

\[ \text{(HW) } \text{let } f : \mathbb{Z}/24 \to \mathbb{Z}/24 \]
be given by \( f(x) = 20x \mod 24 \).

Find Ker \( f \), describe \( (\mathbb{Z}/24)/\text{Ker } f \) and the injective homomorphism
\[ \bar{f} : (\mathbb{Z}/24)/\ker f \rightarrow \mathbb{Z}/24 \]

such that

\[ \mathbb{Z}/24 \xrightarrow{f} \mathbb{Z}/24 \]

\[ \downarrow \]

\[ (\mathbb{Z}/24)/\ker f \xrightarrow{\bar{f}} \mathbb{Z}/24 \]