If $F$ is a field and $p(x)$ is an irreducible polynomial in $F[x]$, we can form a ring of polynomials in one variable $F \subseteq F(x)$.

A field $F(x)$ where $x$ is a root of the polynomial $p(x)$. "We can attach a root of an irreducible polynomial" to a field (First example: how $\mathbb{C}$ is made of $\mathbb{R}$: attach a root $i$ of $x^2 + 1$.)
The construction: \( \mathbb{F} [x] / (\pi(x)) \)

principal ideal of polynomial multiples of \( \pi(x) \)

factor ring.

element are cosets.

John Wiltshire-Gordon’s lectures \( \Rightarrow \mathbb{F} [x] / (\pi(x)) \) a field.

We have a homomorphism \( \mathbb{F} \rightarrow \mathbb{F} [x] / (\pi(x)) = \mathbb{F}(\bar{x}) \).

Friday: a homomorphism of rings whose domain and codomain are fields is always
An injective homomorphism can be treated as an inclusion.

OK, so we have attached a root of \( p(x) \) to \( F \) to form \( F(x) \). This means over \( F(x) \),

\[
( \text{in } F(x)[x] )
\]

\( p(x) \) is no longer reducible. It has, after all, a root, so it has a linear factor. Does this mean that \( p(x) \) factors completely into linear factors in \( F(x) \)? Answer: Not necessarily.
We have one root, but maybe not all the roots. And so, to get all the roots, we need to repeat the procedure.

Example: \( F = \mathbb{Q} \). If \( p(x) \) is quadratic, if we attach one root, we will leave the other one also by long division. \[
\begin{align*}
Q(\sqrt{2}) & = -\sqrt{2} \\
\frac{x - \sqrt{2}}{x + \sqrt{2}} & = \frac{x^2 - 2}{x^2 - 2} = (x + \sqrt{2})(x - \sqrt{2}) \\
\end{align*}
\] So with a quadratic polynomial \( p(x) \), we cannot get
an example where attaching one root would fail to attach all the roots.

Example: let us look at a cubic polynomial

$$x^3 - 2, \quad F = \mathbb{Q}$$

A cubic polynomial which does not have a root in $F$ is irreducible over $F$ (if it factored into factors of lower degree, one factor would have to be linear). \[ x^3 - 2 \text{ is irreducible.} \]

Consider the field \[ \mathbb{Q}(\sqrt[3]{2}). \]

Cotset version: \[ \mathbb{Q}[x]/(x^3 - 2) \]
The elements are:

\[(x)\quad a 2^{2/3} + b 2^{1/3} + c,\quad a, b, c \in \mathbb{Q} \]

the different roots are

\[\{ax^2 + bx + c\},\]

\[a, b, c \in \mathbb{Q}\]

But does \(\mathbb{Q}(\sqrt[3]{2})\) have all the roots of \(x^3 - 2\)?

Does \(x^3 - 2\) factor into linear factors as a polynomial with coefficients in \(\mathbb{Q}(\sqrt[3]{2})\)?

\[\text{No: We know the other roots of } x^3 - 2 \text{ } \to \]

(looking into complex numbers, we know the root of \(x^3 - 2\):

\[3\sqrt[3]{2}, \frac{3\sqrt[3]{2}}{2}, \frac{3\sqrt[3]{2}}{2}(2\sqrt[3]{2})\]

\[\to 120^\circ, 120^\circ, 120^\circ\]
I'm a field

where \( x^3 - 2 \)

splits into three linear factors,

I must also have three third roots of unity \( (1, \xi, \xi^2) \).

But the field \( \mathbb{Q} [\sqrt[3]{2}] \), as we represented \( \xi \) in (\star), is a subfield of \( \mathbb{R} \). In \( \mathbb{R} \), we certainly have only one third root of unity. (That would be a contradiction.)
What if I think of the same field as

\[ \mathbb{Q}(\sqrt[3]{2}) \, ? \]

Element:

\[ a \sqrt[3]{2} (\sqrt[3]{2}) + b \sqrt[3]{2} (\sqrt[3]{2}) + c \, , \quad a, b, c \in \mathbb{Q}. \]

A different field because it is no longer a

subfield of \( \mathbb{R} \) !!!

But of course it is homomorphic !!!

\[ \mathbb{Q}(\sqrt[3]{2}) \rightarrow \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{2}) \]
$\sqrt[3]{2} \xrightarrow{} \mathbb{Q}_3 \sqrt[3]{2}$

These questions are really explained if I keep adding roots until I have all the roots of the original polynomial:

$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{Q}(\sqrt[3]{2}, i\sqrt[3]{2}) \xrightarrow{\text{degree } 3} \mathbb{Q}(\sqrt[3]{2}, i\sqrt[3]{2}, i\sqrt[3]{3}) \xrightarrow{\text{degree } 2} \mathbb{Q}(\sqrt[3]{2}, i\sqrt[3]{2}, i\sqrt[3]{3}, \sqrt[3]{3})$ 

$6$-dimensional vector space over $\mathbb{Q}$

(we say: has degree $6$)
The field $\mathbb{Q}(\sqrt[3]{2}, i, \sqrt[5]{3})$ has automorphisms which permute the roots $\sqrt[3]{2}$, $\xi_2, (\sqrt[3]{2})^2, (\xi_3)^2$.

In this case, any permutation of the roots gives rise to an automorphism. This group is called the Galois group \( \text{Gal}(\mathbb{Q}(\sqrt[3]{2}, i, \sqrt[5]{3})/\mathbb{Q}) \approx \Sigma_3 \) where the symmetric group on 3 elements.

\[ p(x) = \frac{x^3 - 2}{x - \sqrt[3]{2}} \]

Compute the polynomial
(It should have real coefficients) and construct an isomorphism of fields

\[ \mathbb{Q}(\sqrt{2})[x]/(p(x)) \cong \mathbb{Q}(\sqrt{2})[y]/(y^2 + 3). \]

(To solve it, "express \( x \) in terms of \( y \),

\( y \) in terms of \( x \)."

Use the above example.)