\[ \mathbb{Q}(\sqrt{2})[x]/(x^2 - 2) \cong \mathbb{Q}(\sqrt{2})[y]/(y^2 + 2) \]

\[ p(x) = \frac{x^2 - 2}{x - \sqrt{2}} \]

Choose a root here: \([x]\)

Roots of \(p(x)\) are \(\sqrt{2}, -\sqrt{2}\)

\[ \begin{align*}
  \xi_3 &= \frac{-1}{2} + \frac{i\sqrt{3}}{2} \\
  \xi_3 &= \frac{-1}{2} - \frac{i\sqrt{3}}{2}
\end{align*} \]
\[ \sqrt[3]{2} = -\frac{\sqrt[3]{2}}{2} \quad \text{[x]} \]

\[ \text{[x]} = -\frac{\sqrt[3]{2}}{2} - \frac{\sqrt[3]{2}}{2} \cdot [y] \]

\[ \mathbb{Q}(\sqrt[3]{2}) \xrightarrow{\varphi(x)} \mathbb{Q}(\sqrt[3]{2})[y]/(y^3 + 3) \]

\[ \text{[x]} \xrightarrow{\varphi} -\frac{\sqrt[3]{2}}{2} - \frac{\sqrt[3]{2}}{2} \cdot [y] \]

Use the homomorphism theorem and the fact that ...
[x] is a root of \( p(x) \)

To go backwork:

\[
[x] = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot [y]
\]

solve for [y]:

\[
[x] + \frac{\sqrt{2}}{2} = -\frac{3\sqrt{2}}{2} \cdot [y]
\]

\[
-\frac{2}{\sqrt{2}} [x] - 1 = [y]
\]

On the last:

\[
-\frac{2}{\sqrt{2}} [x] - 1 \rightarrow \text{exam!}
\]

Continuing this example: This field that
we just discussed could be called \((Q(\sqrt[3]{2})) (\xi, \sqrt[3]{2}) = \)

\[= Q(\sqrt[3]{2}, \xi, \sqrt[3]{2}) \]

A less awkward notation

I attached all the roots of the polynomial

\[x^3 - 2 \] in order

This is called a splitting field for the polynomial \(x^3 - 2\).

The splitting field \(Q(\sqrt[3]{2}, \xi, \sqrt[3]{2}) = E\) has a lot of automorphisms\( \cong (\text{isomorphisms to itself}) \)
All automorphisms of $E$ will act by permutation on the roots of $x^3 - 2: \{\sqrt[3]{2}, \zeta_3\sqrt[3]{2}, \zeta_3^2\sqrt[3]{2}\}$.

Example: Construct a non-trivial automorphism ($\neq \text{Id}$) of $E$ which leaves $\mathbb{Q}(\sqrt[3]{2})$ fixed.

$f : E \rightarrow E$  
For every $x \in \mathbb{Q}(\sqrt[3]{2})$, $f(x) = x$.

Think of $[x]$ as $\zeta_3\sqrt[3]{2}$.

$$\mathbb{Q}(\sqrt[3]{2})[x]/(\rho(x)) \rightarrow \mathbb{Q}(\sqrt[3]{2})[x]/(\rho(x))$$

$E$ 

Think of $E$ 

This way: $[x]$
\[ \sqrt[3]{2} \rightarrow \sqrt[3]{2} \]

Express in terms of \( \sqrt[3]{2} \)

\[ \sqrt[3]{2} \cdot \sqrt[3]{2} = \left( \frac{\sqrt[3]{2}}{\sqrt[3]{2}} \right)^2 \]

\[ [x] \rightarrow \frac{[x]^2}{\sqrt[3]{2}} \]
A forced (it must be a permutation)

Example: What if I start with $Q(\sqrt[3]{2})$ instead of $Q(\sqrt{2})$? I can think of $E$ as

$$E = \frac{Q(\sqrt[3]{2})[z]}{(g(z))} \quad \xrightarrow{g} \quad \frac{Q(\sqrt[3]{2})[z]}{(g(z))}$$

$$g(z) = \frac{z^3 - 2}{z - \sqrt[3]{2}}$$

Roots of $g(z): \sqrt[3]{2}, \sqrt[3]{2^2}, \sqrt[3]{2^3}$

Choose $2$
\begin{align*}
\frac{e}{r} &= \frac{\frac{5\sqrt{12}}{2}}{\sqrt{\frac{12}{7}}} \\
\frac{2}{5\sqrt{2}} &= \frac{\frac{5\sqrt{12}}{2}}{\sqrt{\frac{12}{7}}} \\
\frac{e}{r} &= \frac{\frac{5\sqrt{12}}{2}}{\sqrt{\frac{12}{7}}} \\
\frac{2}{5\sqrt{2}} &= \frac{\frac{5\sqrt{12}}{2}}{\sqrt{\frac{12}{7}}} \\
\text{Express as a polynomial of } \frac{e}{r}, \frac{2}{5\sqrt{2}}
\end{align*}
The group of automorphisms of $E$ (splitting field of $x^3 - 2$) is identified with the group of permutations isomorphic in a given way on the set of roots of $x^3 - 2$.
Galois theory tells us

how to use the group $\Sigma = \text{Gal}(E/\mathbb{Q})$


to find all fields $F$ such that

$\mathbb{Q} \subseteq F \subseteq E$.

The recipe: Take a subgroup $H \subseteq \Sigma$, take

the field $E^H = \{ \text{all elements } x \text{ of } E \text{ which are not moved by } H \}$.

$\forall h \in H \Rightarrow h(x) = x$. \underline{fixed}
Examples:

\[ \mathbb{Z}/2 \cong H = \left\{ \sqrt{2}, \sqrt[3]{2}, \sqrt{3}\sqrt{2}, \text{Id} \right\} \subset \mathbb{S} \]

\[ E = \mathbb{Q}(\sqrt[3]{2}) \]

HW0:
We can get the subfields \( \mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\sqrt{3}\sqrt{2}) \) in a similar way. What are the corresponding subgroups of \( \mathbb{S} \)?

What about \( \mathbb{Z}/3 \cong \mathbb{Z} = \text{cyclic subgroup generated by} \)

\[ \sqrt{2} \]

\[ \sqrt[3]{2}, \sqrt{3}\sqrt{2}, \sqrt[3]{2} \]
What is $E^r$?

$E^r = \mathbb{Q}(i\sqrt{2}) = \mathbb{Q}(\sqrt{2})$

Reason: $\sqrt{2} = \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}}$.

[Text circled]

Use the methods in today's examples to construct an isomorphism

$h: \mathbb{Q}(\sqrt{2}) [x] / (p(x)) \rightarrow \mathbb{Q}(\sqrt{2}) [x] / (q(x))$

relation as above

$p(x) = \frac{x^3 - 2}{x - \sqrt{2}}$

$q(x) = \frac{x^3 - 2}{x - \sqrt{2}}$
\[ \frac{3}{2} \quad \frac{3}{2} \quad \frac{3}{2} \quad \sqrt{2} \]

**Hint**: There are two solutions. Pick what he should do on the remaining roots, then express it as expressing the image of \([x]\) in terms of \([z]\) and \(\frac{3}{2} \sqrt{2}\).