S = Sample space

E ⊆ S  
in a measurable event:

Probability: \( P(E) \)

\[ 0 \leq P(E) \leq 1. \]

\[ P(S) = 1. \]

If \( E_1, E_2, E_3, \ldots \) are measurable events, \( E_i \cap E_j = \emptyset \) \( \forall \ i \neq j \),
then
\[ P(E_1 \cup E_2 \cup \ldots) = \sum \limits_{i=1}^{\infty} P(E_i). \]
Example: Equally likely outcomes in light of the axioms:

\[ S = \{1, 2, 3, 4, 5, 6\} \]

All events are measurable (usually the case for finite sample space)

\[ P(E) = \frac{|E|}{6} \quad \left( |E| \text{ is the number of elements in } E \right) \]

2.9.

\[ E_1 = \{1, 2\} \quad E_2 = \{3, 4\} \quad E_1 \cap E_2 = \emptyset \]

\[ P(E_1) = \frac{2}{6} = \frac{1}{3} \quad P(E_2) = \frac{2}{6} = \frac{1}{3} \]
\[ P(E_1 \cup E_2) = P(\{1, 2, 3, 4, 5\}) = \frac{5}{6} = \frac{2}{3} = P(E_1) + P(E_2). \]

**Generally:** If \( S \) is a finite sample space:

**Equally likely outcomes scenario:**

\[ P(E) = \frac{|E|}{|S|}. \]

**Example:** A biased die.

\[ P(1) = a_1, \quad P(2) = a_2, \quad \ldots, \quad P(6) = a_6 \]

\[ 0 \leq a_i \leq 1, \quad a_1 + a_2 + a_3 + \ldots + a_6 = 1 \]
e.g. \( a_1 = \frac{1}{12} \), \( a_2 = \frac{1}{12} \), \( a_3 = \frac{1}{12} \), \( a_4 = \frac{1}{12} \), \( a_5 = \frac{1}{12} \), \( a_6 = \frac{7}{12} \).

\[
P(1, 2) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}
\]

\[
\begin{array}{c}
\uparrow \\
\uparrow \\
a_1 \quad a_2
\end{array}
\]

---

**Generally:** If finite. Then to each \( s \in S \) we assign a probability \( a_s \), \( 0 \leq a_s \leq 1 \),

Condition: \( \sum_{s \in S} a_s = 1 \).

\[
P(E) = \sum_{s \in E} a_s
\]
Typical infinite examples are based on volume. (in any dimension)

Example: $S = \{(x, y) | 0 \leq x \leq 1\} \times \{(x, y) | 0 \leq y \leq 1\}$ = the unit interval.

Imagine: I throw a dart which will lands, (say), with equal likelihood anywhere in the interval (at one point).

$P(x = f) = 0$.

one particular number
\[ p([a,b]) = b - a \]

\[ \text{length of the interval.} \]

\[ p((a,b)) = p([a,b]) = p((a,b)) = p([a,b]) = b - a. \]

**Example (the Cantor set):**

\[ K_0 = [0,1] \]

\[ K_1 = [0, \frac{1}{3}] \cup \left[ \frac{2}{3}, 1 \right] \]

\[ K_2 = [0, \frac{1}{9}] \cup \left[ \frac{1}{3}, \frac{2}{9} \right] \cup \left[ \frac{2}{3}, \frac{7}{9} \right] \cup \left[ \frac{8}{9}, 1 \right] \]

\[ \vdots \]

**Recipe:** At each stage, remove from each interval the middle third.
$K_n = 2^n$ intervals of length $\frac{1}{3^n}$ each.

$K = \bigcap_{n=1}^{\infty} K_n$.

$P(K) = \lim_{n \to \infty} P(K_n) = \lim_{n \to \infty} \frac{2^n}{3^n} = 0$.

Not all sets can be measurable in this example! The length-defined probability is called Lebesgue measure.
Principle of inclusion and exclusion:

How to calculate $P(E_1 \cup E_2)$ when they are not (necessarily) disjoint.

Given $P(E_1)$, $P(E_2)$, $P(E_1 \cap E_2)$.

\[ E_1 \cap E_2 = E_1 \cap E_2^c \]
\[ E_2 \cap E_1 = E_2 \cap E_1^c \]

\[ P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \]
More detailed proof:

\[ E_1 = (E_1 \cap E_2) \cup (E_1 \cap E_3) \]
\[ (E_1 \cap E_2) \cap (E_1 \cap E_3) = \emptyset \]

\[ P(E_1) = P(E_1 \cap E_2) + P(E_1 \cap E_3) \quad (1) \]

Similarly,

\[ P(E_1) = P(E_2 \cap E_1) + P(E_3 \cap E_1) \quad (2) \]

\[ P(E_1 \cup E_2) = P(E_1 \cap E_2) + P(E_2 \cap E_1) + P(E_4 \cap E_2) \quad (3) \]

\[ (1) \land (2) \land (3) \implies (4) \]

Example: In a year, there are 25 students.
14 take chemistry, 9 take physics, 3 take both.
chemistry and physics. If I choose a student with equal likelihood at random, what is the probability that he/she is taking either chemistry or physics or both?

\[ \text{Solution: } P(P) = \frac{9}{25}, \quad P(C) = \frac{14}{25}, \quad P(P \cap C) = \frac{3}{25}. \]

\[ P(P \cup C) = \frac{9}{25} + \frac{14}{25} - \frac{3}{25} = \frac{20}{25} = \frac{4}{5}. \]

Example: Suppose a student does take physics and chemistry. Suppose the probability of passing at least
one of these two courses is 0.9, the probability of passing chemistry is 0.7, probability of passing physics is 0.8. What is the probability of passing both?

Solution:

\[ P(P \cup C) = 0.9 \]

\[ P(C) = 0.7 \]

\[ P(P) = 0.8 \]

\[ P(P \cup C) = P(C) + P(P) - P(P \cap C) \]

\[ 0.9 = 0.7 + 0.8 - P(P \cap C) \]
\[ P(P \cap C) = 0.7 + 0.8 - 0.9 = 0.6. \]

Inclusion and exclusion for 3 events:

\[ P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - \]

\[ P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3). \]
The general case ($k$ events):

$$P(E_1 \cup \cdots \cup E_k) = P(E_1) + \cdots + P(E_k)$$

$$- P(E_1 \cap E_2) - P(E_1 \cap E_3) - \cdots - P(E_{k-1} \cap E_k)$$

$${k \choose 2} \text{ summands}$$

$$+ P(E_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_4) + \cdots + P(E_{k-2} \cap E_{k-1} \cap E_k)$$
\binom{k}{3} \text{ terms}

\[ + (-1)^{k-1} \mathbb{P}(E_1 \cap \cdots \cap E_k). \]

The \( r \)-th term:

\[ (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq k} \mathbb{P}(E_{i_1} \cap \cdots \cap E_{i_r}). \]

Proof in terms of "over counting and under counting".
Suppose an odd come \( x \) is in precisely \( r \) of the events \( E_1, \ldots, E_k \). How many times \( x \) counted in the formula \( \Theta \)?

\[
\begin{align*}
r - \binom{r}{2} + \binom{r}{3} - \cdots + (-1)^{r-1} \binom{r}{r}^2 &= 1 \\
1 - r + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} &= \quad \end{align*}
\]
\[ \sum_{i=0}^{r} \binom{r}{i} (-1)^i = ((-1) + 1)^r = 0. \]

\[ (x+y)^r = \sum_{i=0}^{r} \binom{r}{i} x^i y^{r-i}. \]

\[ x = -1, \quad y = 1 \]

This is why each outcome is counted exactly once!

**Numerical example:**

\[ S = \text{squash}, \]
\[ B = \text{badminton}, \]
\[ T = \text{tennis} \]

\# of people playing
Then we have a class:

| 36 | T | 22 | T ∩ S |
| 28 | S | 12 | T ∩ B |
| 18 | B | 9  | S ∩ B |

4 students.

How many people play at least one of the sports T, S, B?

**Solution:**

\[
|T| + |S| + |B| - |T \cap S| - |T \cap B| - (S \cap B) + |S \cap B \cap T|
\]

\[
= 36 + 28 + 18 - 22 - 12 - 9 + 4 = 43
\]
Harder example in probability:

Ex. 56 on p. 34

3 balls are randomly drawn from a bowl containing 6 white and 5 black balls. What is the probability that 1 ball is white and two are black?

*) without replacement.

Solution: Sample space \( S \) of subsets of 3 balls among
The set of all 11 balls

\[ |S| = \binom{11}{3} = \frac{11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3} \]

\[ E = \{ \text{subset of 1 among the 6 white balls, } \text{subset of 2 among the 5 black balls} \} \]

\[ |E| = \binom{6}{1} \cdot \binom{5}{2} \]

\[ P(E) = \frac{|E|}{|S|} = \frac{\binom{6}{1} \binom{5}{2}}{\binom{11}{3}} \]
Note: This solution was "a little clever."

A cleverer solution: which ball did I choose first? Sample space $S$:

- Ball 1
- Ball 2

$|S| = 11 \cdot 10 - 9$

We could model the situation by the sample space $S$ instead of $S'$, because the outcomes still capture all the information we are recording, and are still equally likely.

This trick is called sample space reduction.
This example in general terms:

Suppose we have \( m \) black balls and \( n \) white balls. Suppose we draw \( k + l \) balls (at equal likelihood, without replacement). What is the probability that \( k \) balls are black and \( l \) balls are white?

\[
S = \text{sets of } k + l \text{ balls among the } m + n
\]

\[
|S| = \binom{m+n}{k+l}.
\]
\[ E = \{k \text{ balls among } m\} \times \{l \text{ balls among } n\} \]

\[ |E| = \binom{m}{k} \binom{n}{l} \]

\[ P(E) = \frac{\binom{m}{k} \binom{n}{l}}{\binom{m+n}{k+l}} \]

Example: Same problem, but with replacement.

Suppose I have \( m \) black balls and \( n \) white balls.
I will draw a ball, record its color, and throw it back. I will do this a total of \( k + l \) times. What is the probability that the color of the ball was black \( k \) times and white \( l \) times?

**Solution:** Sample space \( S \); sequences of \( k + l \) elements in the set of \( m + n \) balls.

\[
|S| = (m + n)^{k+l}
\]
The event $E$: sequences of $k+l$ elements in the set of $m+n$ balls where $k$ of the terms of the sequence where black, $l$ were white.

$$|E| = \binom{k+l}{k} \frac{k}{m} \frac{l}{n}$$

$$P(E) = \left( \binom{k+l}{k} \frac{k}{m} \frac{l}{n} \right)^{k+l} = \left( \frac{k+l}{k} \frac{m}{m+n} \right)^{k} \left( \frac{n}{m+n} \right)^{l}$$
\[
\frac{m}{m+n} = \text{probability that one randomly chosen ball is black}
\]

\[
\frac{n}{m+n} = \text{probability that one randomly chosen ball is white}
\]

Example 5j, p. 38: A deck of 52 playing cards is shuffled. Turn cards over one at a time until the first ace appears. Is the next card more likely to be the ace of spades \(\spadesuit\) or the two of clubs? \(\clubsuit\)
Solution:

![Card Diagram]

the special card \( \heartsuit \)

Modify the experiment by taking the card \( \heartsuit \) out of the deck, shuffling the deck without \( \heartsuit \), and then placing \( \heartsuit \) back in randomly.

What is the probability that \( \heartsuit \) will be right after the first ace in the deck shuffled without \( \heartsuit \)?

Answer: \( \frac{1}{52} \). (It does not matter)
(1) Suppose in a class of people, there are 4 subjects to take: \( M = \text{math} \)
\( E = \text{English} \)
\( C = \text{chemistry} \)
\( P = \text{physics} \)

Suppose the numbers of people taking the subject are as follows:

\( M : 10 \quad E : 12 \quad C : 9 \quad P : 8 \)

\( M \cap E : 8 \quad M \cap C : 6 \quad M \cap P : 4 \quad E \cap C : 6 \quad E \cap P : 5 \)
C & P = 4 nobody took more than 2 subjects.

How many people took at least one subject?

2. Suppose we have a hat with 10 red balls and 8 blue balls. Suppose I choose 6 balls without replacement. What is the probability that 4 will be red and 2 will be blue?

3. Same problem, but with replacement.