Material on exam: up to Bayes formula

More examples on independent trials

Example 42 p. 86

We perform independent trials which result in success with probability $p$ and failure with probability $1-p$.

What is the probability that $n$ successes occur
Let there be at least $m$ failures.

Solution: Idea: look at the first $m + m - 1$ trials.

In them, there will be either at least $m$ successes or at least $m$ failures, but not both.

$$m - 1 + m - 1 = m + m - 2 < m + m - 1$$

$$m + m = m + m > m + m - 1$$

$$E = \{ \text{there are at least } m \text{ successes in the first } m + m - 1 \text{ trials} \}$$

$$E = E_m + E_{m+1} + \cdots + E_{m+m-1}$$
\[ E_k = \left\{ \text{there are exactly } k \text{ successes in the first } m + n - 1 \text{ trials} \right\} \]

\[ P(E_k) = \binom{m + n - 1}{k} p^k (1 - p)^{m + n - k - 1} \]

\[ P(E) = \sum_{k=m}^{m+n-1} \binom{m + n - 1}{k} p^k (1 - p)^{m + n - k - 1} \]

Example: Suppose in a game, teams A and B are equally likely to win. What is the probability that
Team A will win a best of 7 series in 6 or fewer games.

Solution: Team A wins: “success”

Team B wins: “failure”

Let 4 successes occur before 3 failures:

\[ \sum_{k=4}^{6} \binom{6}{k} \frac{1}{2^6} = \frac{(15 + 6 + 1)}{2^6} = \frac{22}{64} = \frac{11}{32} \]
Example 4b: Two gamblers A, B bet on successive outcomes of flipping a coin. \( P(H) = p \), \( P(T) = 1-p \). If the coin comes up H, A collects 1$ from B. If the coin comes up T, B collects 1$ from A. The game is over when A or B is out of cash. Suppose A starts with i$ and B starts with \((N-i)\)$$. What is the probability that A wins everything?

Solution: \( N = \text{constant} \).
\[
\begin{align*}
P_i &= P(E_i) \\
E_i &= \text{A wins after starting with i$}.
\end{align*}
\]
Let $H$ be the first coin come up heads.

Bayes' formula:

$$
P_i = P(E_i) = \frac{P(E_i \mid H) \cdot P(H)}{P(E_i)} + \frac{P(E_i \mid H^c) \cdot (1 - P(H))}{P(E_i)}
$$

$$
= P_{i+1} \cdot p + P_{i-1} \cdot (1-p)
$$

A linear recursion.
\[ p_N - p_{N-1} = \left( \frac{1-p}{p} \right)^{N-1} p_1 \]

\[ p_\cdot\cdot - p_1 = \left[ \frac{1-p}{p} + \frac{(1-p)^2}{p^2} + \cdots + \frac{(1-p)^{i-1}}{p^{i-1}} \right] p_1 \]

\[ p_\cdot\cdot = \left[ 1 + \frac{1-p}{p} + \frac{(1-p)^2}{p^2} + \cdots + \frac{(1-p)^{i-1}}{p^{i-1}} \right] p_1 \]

If \( p = \frac{1}{2} \)

\[ p_\cdot\cdot = n \cdot p_1 \quad \otimes \]
If \( p \leq \frac{1}{2} \):

\[
p_i = \frac{1 - \left(\frac{1-p}{p}\right)^i}{1 - \frac{1-p}{p}}
\]

For \( i = N \) we know \( p_N = 1 \).

If \( p > \frac{1}{2} \) and \( 1 = p_N = N p_1 \):

\[
p_1 = \frac{1}{N}
\]

\[
p_i = \frac{n_i}{N}
\]

If \( p \geq \frac{1}{2} \):

\[
p_N = \frac{1 - \left(\frac{1-p}{p}\right)^N}{1 - \frac{1-p}{p}}
\]
If the game is unbiased, then our chances of winning are directly proportional to the amount of money.

If it is not fair, it becomes exponential.

\[ P(\mathbb{Z} | \mathcal{F}) \]

Conditional probability of a probability (on the sample space \( \mathcal{F} \))
Example 56: A female chimpanzee has given birth, but it is not certain which of two male chimpanzees is the father.

Before DNA analysis:

\[ \text{Probability } p \]

\[ \text{Probability } 1-p \]

DNA analysis: One section of genome:

Mother: (A, A)

Male 1: (a, a)

Male 2: (A, a)

Baby: (A, a).
What is the probability that Male 1 is the father?

Solution: Conditional on the event that

Mother: \( (A_A) \)

Male 1: \( (C_a,a) \)

Male 2: \( (A_i,a) \)

\[ P(M_1 \mid B_{A_a,a}) = \frac{P(M_1 \cap B_{A_a,a})}{P(B_{A_a,a})} = \]
\[
\begin{align*}
\frac{P(B_{A \alpha} | M_1) \cdot P(M_1)}{P(B_{A \alpha} | M_1) \cdot P(M_1) + P(B_{A \alpha} | M_2) \cdot P(M_2)} & = \frac{1 \cdot p}{1 \cdot p + \frac{1}{2} (1-p)} \\
& = \frac{2p}{1+p}
\end{align*}
\]

\[p < 1 \quad \text{for small } p = 1 \text{ equal} \]
\[
\frac{2p}{1+p} \quad \text{vs.} \quad p \\
2p \quad \text{vs.} \quad p^2 + p
\]

\[
2 \quad \text{vs.} \quad p + 1
\]

\[
\frac{2p}{1+p} > p \quad \iff \quad 2 > p + 1
\]

By Bayesian analysis, the DNA of the baby matching M2 increases the probability that M1 is the father.
Example 5e: Laplace's rule of succession:

There are k+1 coins in a box. When flipped, the i-th coin turns up head with probability \( \frac{i}{k} \).

A coin is randomly selected and then repeatedly flipped. If the first n flips are all heads, what is the conditional probability that the (n+1)th flip will also be heads?

Solution: \( E_i = \{i\text{th coin}\} \)

\( F_n = \{\text{first } n \text{ flips are H}\} \).
\[ H = \{ (x_i+1)^{st} fC_i u \rightarrow H \} \]

using: conditional probability is a probability

\[ P(H|F_m) = \sum_{i=0}^{k} P(H|F_m \cap C_i) \cdot P(C_i|F_m) \]

\[ P(H|F_m \cap C_i) = \frac{\lambda_i}{k} \]

\[ F_m \text{ irrelevant} \]

\[ P(C_i|F_m) = \frac{P(C_i \cap F_m)}{P(F_m)} = \frac{P(F_m|C_i)P(C_i)}{\sum_{i=0}^{k} P(F_m|C_i)P(C_i)} = \]
\[
\begin{align*}
\sum_{i=0}^{k} \left( \frac{a_i/k}{k+1} \right)^m &= \frac{\left( \frac{a_i/k}{k+1} \right)^m}{\sum_{i=0}^{k} \left( \frac{a_i/k}{k+1} \right)^m} \\
&= \frac{\left( \frac{a_i/k}{k+1} \right)^m}{\sum_{i=0}^{k} \left( \frac{a_i/k}{k+1} \right)^m}
\end{align*}
\]

\[
P(H|F_m) = \sum_{i=0}^{k} \frac{i}{k} \cdot P(C_i|F_m) = \frac{\left( \frac{a_i/k}{k+1} \right)^{m+1}}{\sum_{i=0}^{k} \left( \frac{a_i/k}{k+1} \right)^m}
\]

Why in probability is it sometimes better to work continuous than discrete? If \( k \gg 0 \).
\[
\frac{1}{k} \sum_{i=0}^{h} \left( \frac{i}{k} \right)^{m+1} \approx \int_{0}^{x} x^{m+1} \, dx = \left[ \frac{x^{m+2}}{m+2} \right]_{0}^{1} = \frac{1}{m+2} \left( \frac{1^{m+2}}{m+2} \right) = \frac{1}{m+2}.
\]
\[ \frac{1}{k} \sum_{i=0}^{n} \left( \frac{n}{k} \right)^n \approx \int_0^1 x^n dx = \left[ \frac{1}{n+1} x^{n+1} \right]_0^1 = \frac{1}{n+1} \]

\[ P \left( H \mid F_n \right) \approx \frac{n+1}{n+2} \]

Example 5e, p. 95

Independent trials result in success with probability \( p \) and failure with probability \( 1-p \).

Compute the probability that a streak of \( n \)
consecutive successes occurs before a streak of
\[ \text{Solution: } E = \text{a streak of } m \text{ consecutive successes} \]
\[ \text{occurs before a streak of } m \text{ consecutive failures.} \]

Condition on the outcome of the first trial.

Success of 1st trial: \( H \).

\[ P(E) = p P(E|H) + (1-p) P(E|H^c). \]
\( F = \{ \text{trials 2 to } m \text{ are successes} \} \)

\[
P(E|H) = P(E|F \cap H) P(F|H) + P(E|\overline{F} \cap H) P(\overline{F}|H)
\]

\[
P(F) = p^{m-1}
\]

\[
P(E|H) = p^{m-1} + \left( 1 - p^{m-1} \right) P(E|H^c)
\]

\( G = \{ 1 \text{ to } m \text{ are failures} \} \)
\[
\begin{align*}
R(E|H^c) &= R(E|C \cap H^c) R(C|H^c) + R(E|C^c \cup H^c) R(C^c|H^c) \\
&= R(E|H) \left( 1 - (1-p)^{m-1} \right)
\end{align*}
\]
\[
\begin{align*}
    P(E|H) &= p^{m-1} + (1 - p^{m-1}) \cdot P(E|H^c) \\
    P(E|H^c) &= \left(1 - (1-p)^{m-1}\right) \cdot P(E|H)
\end{align*}
\]

\[
\begin{align*}
    P(E|H) &= p^{m-1} + (1 - p^{m-1}) \left(1 - (1-p)^{m-1}\right) \cdot P(E|H)
\end{align*}
\]

\[
\begin{align*}
    P(E(H)) &= \frac{p^{m-1}}{1 - (1-p)^{m-1} \cdot \left(1 - (1-p)^{m-1}\right)}
\end{align*}
\]

\[
\begin{align*}
    P(E(H)) &= \frac{p^{m-1}}{p^{m-1} + (1-p)^{m-1} - p^{m-1}(1-p)^{m-1}}
\end{align*}
\]
\[ p(E|H^c) = \frac{\left(1 - (1-p)^{m-1}\right)p^m}{p^{m-1} + (1-p)^{m-1} - p^{m-1}(1-p)^{m-1}} \]

\[ p(E) = p(E|H) \cdot p + p(E|H^c)(1-p) = \]

\[ = \frac{p^{m-1}(1 - (1-p)^m)}{p^{m-1} + (1-p)^{m-1} - p^{m-1}(1-p)^{m-1}} \]
If team A and team B play a best of 7 playoff series and they are equally likely ($p = \frac{1}{2}$) to win any particular game, and the outcomes of games are independent, what is the probability that there will be a game 7?

2. Same problem, except the probability of team A winning any particular
game is 2/3.

3. A couple has 2 children. What is the probability that both are boys assuming the younger one is a boy?

4. If team A has probability \( \frac{3}{4} \) of winning against team B, what is the probability that team A will have an all-time 3 game winning streak?
against B before B ever had a 3 game winning streak against A?

Tomorrow exam: 1:00 - 3 PM in class.