Problem 2: Bridge: 52 cards

(a) What is the probability that $E$ and $W$ are dealt all the aces? (in one game)

$S = \{ \text{cards} \}$ $E \cup W$ get together

$|S| = \binom{52}{26}$

$E = \{ E \text{ and } W \text{ get } 4 \text{ aces} \}$

$|E| = \binom{48}{22}$

$P(E) = \dfrac{\binom{48}{22}}{\binom{52}{26}}$
(b) when 26 by 13:

\[
\begin{pmatrix}
48 \\ 9 \\
\hline
52 \\ 13
\end{pmatrix}
\]

Problem 3: \( \text{P (3 of a kind in poker)} \)

Hand: 5 cards out of 52

\[ a \ a \ a \ b \ c \]

\[ a \neq b \neq c \neq a \]

\[ C \{ 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A \} \]
\[ |S| = \binom{52}{5} \]

\[ E = \{ 3 \text{ of a kind} \} \]

\[ \begin{array}{c}
\text{chance of a} \\
\text{suit of } 5, c
\end{array} \]

\[ \begin{array}{r}
\text{choice of } a \\
\text{choices of } \{ 5, c \}
\end{array} \]

\[ \begin{array}{c}
(13 \cdot (4)) \cdot \binom{12}{2} \cdot 4 \cdot 4
\end{array} \]

\[ P(E) = \frac{13 \cdot (4) \cdot 12 \cdot 4 \cdot 4}{\binom{52}{5}} = \frac{13 \cdot 4 \cdot 12 \cdot 11 \cdot 4 \cdot 2}{\binom{52}{5}} \]

Next Midterm: June 3

Quiz: This Tuesday 5/26
Random variables

A random variable $X$ is a measurable function on the sample space $S$:

$$X : S \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$$

Measurable function means:

$$E_x = \{ s \in S \text{ such that } X(s) \leq x \}$$
\( P(E_x) = P\{X \leq x\} \)

is called the cumulative distribution of the variable \( X \).

1. A simple example: Say I play a game where I flip a coin. If \( H \), I win $10, if \( T \), I lose $10.

Write down the random variable, and calculate its cumulative distribution.
$y$ is the image of $X \leq y$.

$x, y \in \mathbb{R}$
A discrete random variable is a random variable $X$ which takes on only at most countably many values.

Example: Suppose I am flipping a coin.

$X$ = how many trials I need
for the first time the coin comes up heads

The values are 1, 2, 3, 4, 5, 6, ... , oo 

\[ S = \{ \text{infinite sequence of heads/tails} \} \]

\[
\begin{align*}
&\text{(H H H H \ldots )} \\
&\text{(T H H \ldots )} \\
&\text{(H T H H \ldots )} \\
&\text{(T T \ldots )} \\
&\overbrace{\text{TT \ldots }}^{m-1} \overbrace{\text{TH (don't care afterward)}}^1
\end{align*}
\]
\[ P(X = n) = \frac{1}{2^n} \]

A cumulative distribution only depends on the probability of individual values arising (the \( x \) probabilities must add up to 1).

**Probability mass function**

**Cumulative distribution from probability mass function:**

\[ P(X \leq m) = \sum_{m \leq m} P(X = m) \]
In the example,

\[ P(X \leq m) = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^m} = 1 - \frac{1}{2^m} \left( \frac{2^m - 1}{2^m} \right) \]

\[
\frac{1}{2} \quad \frac{3}{4} \quad \frac{7}{8} \quad \frac{15}{16} \quad \cdots
\]

Cumulative distribution graph:

\[ y \quad \frac{1}{4} \quad \frac{7}{8} \quad \cdots \]

\[ 1 \quad 2 \quad 3 \quad 4 \quad \cdots \]
(Geometric variable)

Comment:

\[ P(X = \infty) = 1 - \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \right) = 0. \]

Statistics: Largely, computing numerical quantities out of the cumulative distribution (alternatively, from the probability mass function).

Example:
Expectation (expected value): (the average, weighted by probability)

\[ E(X) = \sum_{x : P\{X = x\} > 0} x \cdot P\{X = x\} \]

In Example 1:

\[ E(X) = 10 \cdot \frac{1}{2} + (-10) \cdot \frac{1}{2} = 0 \]

In Example 2:

\[ E(X) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \cdots \]
\[
\frac{1/2}{(1 - 1/2)^2} = \frac{1/2}{1/4} = 2
\]

**Method:** Set \( x = \frac{1}{2} \)

\[
1 - x + 2x^2 + 3x^3 + 4x^4 + \cdots =
\]

\[
= x \left( 1 + 2x + 3x^2 + 4x^3 + \cdots \right)
\]

\[
= x \left( \frac{x}{1 - (x)} \right) = \frac{x}{1 - x}
\]

\[
\frac{1}{1 + x + x^2 + \cdots} = \frac{1}{\frac{x}{1 - x}} = \frac{1 - x}{x}
\]

\[
\frac{1}{-\frac{x}{1 - x}} = \frac{1 - x}{x}
\]

\[
\lim_{x \to 0} \frac{x + 1}{x} = 1
\]

\[
\lim_{x \to \infty} \frac{1 + x + x^2 + \cdots}{x} = \frac{1}{1 - x}
\]
\[ \frac{x}{(1-x)^2} \]

\[ f = x \quad (\frac{x}{1-x})' = \frac{1-x + x}{(1-x)^2} = \frac{1}{(1-x)^2} \]

\[ g = 1-x \]

\[ (\frac{f}{g})' = \frac{f'g - fg'}{g^2} \]

Comment on existence of expectation.

Expectation may not exist.
Modification of Example 2:

In round 1, player A flips a coin.
If the coin comes up H, A wins $1 from player B. If not, ...

In round 2, A hands the coin to player B.
But the stake double. If it comes up H, B wins $2 from A. If not, ...

In round 3, A gets the coin again. The stake double in each round.
Once one of the players wins, the game is over.

\[ X = \text{winnings of player A.} \]

(possibly negative)

\[ E(X) = 1 \cdot \frac{1}{2} + (-2) \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + (-8) \cdot \frac{1}{16} + \ldots = \]

\[ = \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \ldots \]

undefined!

Comment: In general, the expectation is only defined
if the series converge absolutely (the sum of absolute values < ∞).

Ex. 4a p.128 X be a discrete random variable.

Probability mass function

\[ P \{X = -1\} = 0.2 \]
\[ P \{X = 0\} = 0.5 \]
\[ P \{X = 1\} = 0.3 \]

Calculate \( E(X) \) , \( E(X^2) \).
Solution: Value: \{-1, 0, 1\}

\[ E(X) = (-1) \cdot 0.2 + 0 \cdot 0.5 + 1 \cdot 0.3 = 0.1 \]

\[ E(X^2) = 1 \cdot 0.2 + 0 \cdot 0.5 + 1 \cdot 0.3 = 0.5 \]

\[ E(X^2) = (E(X))^2 \]

Note: If \( X, Y \) are two random variables on the same sample space, then jointly distributed random variables.

\[ E(X + Y) = E(X) + E(Y) \]

If the right-hand side is defined.
Example: Let $X$ satisfy

$$P\{X = x_0\} = 1 \quad \text{for some value } x_0.$$ 

$$E(X) = x_0$$

$$E(X^2) = x_0^2 = (E(X))^2.$$ 

(This is the only case when that happens.)

We define the **variance**

$$\text{var}(X) = E(X^2) - (E(X))^2.$$
The **standard deviation**

\[ \sigma(X) = \sqrt{\text{var}(X)}. \]

(Warning: there are other meanings of the term!)

**Proposition:**

\[ \text{var}(X) = E \left( \left( X - E(X) \right)^2 \right) \]

**Proof:**

\[ E \left( \left( X - E(X) \right)^2 \right) = \]

**Note:**

\[
\begin{align*}
E(X - E(X)) &= E(X) - E(E(X)) \\
&= E(X) - E(X) = 0
\end{align*}
\]
\[
\begin{align*}
&= \mathbb{E} \left( X^2 - 2X \mathbb{E}(X) + \mathbb{E}(X)^2 \right) = \\
&= \mathbb{E}(X^2) - 2\mathbb{E}(X)\mathbb{E}(X) + \mathbb{E}(X)^2 = \mathbb{E}(X^2) - \mathbb{E}(X)^2, \quad \square
\end{align*}
\]

In particular: \[\text{var}(X) \geq 0.\]

\[\text{var}(X + b) = \text{var} X\]

\[\uparrow\]

\[\text{constant}\]

\[\begin{align*}
\text{var}(X + b) &= \mathbb{E} \left( (X + b - \mathbb{E}(X + b))^2 \right) = \\
&= \mathbb{E} \left( (X + b - (\mathbb{E}(X) + b))^2 \right) = \mathbb{E}((X - \mathbb{E}(X))^2). \quad \square
\end{align*}\]
Example: Compute the variance of the random variable $X$ of Ex. 4a p. 126:

\[ P\{ X = -1 \} = 0.2 \]
\[ P\{ X = 0 \} = 0.5 \]
\[ P\{ X = 1 \} = 0.3 \]

Solution:

\[ E(X^2) = 1 \cdot 0.2 + 0 \cdot 0.5 + 1 \cdot 0.3 = 0.5 \]
\[ E(X) = (-1) \cdot 0.2 + 0 \cdot 0.5 + 1 \cdot 0.3 = 0.1 \]
\[ E(X)^2 = 0.01 \]
\[ \text{var}(X) = 0.5 - 0.01 = 0.49. \]

Concrete systematic classes of discrete random variables, studied by their cumulative distributions.

The Bernoulli variable \( \leq \) just an event

\[ P \{ X = 1 \} = p \]

\[ P \{ X = 0 \} = 1 - p. \]
**Expected value:**

\[ E(X) = 1 \cdot p + 0 \cdot (1-p) = p. \]

**Calculating variance:**

\[ E(X^2) = 1 \cdot p + 0 \cdot (1-p) = p. \]

\[ \text{var}(X) = E(X^2) - E(X)^2 = p - p^2 = p(1-p). \]

The binomial variable = sum of \( n \) independent jointly distributed Bernoulli variables.
$X_1, \ldots, X_n$ are independent if $P(X_1 \leq x_1, \ldots, P(X_n \leq x_n)$ are independent for any values of $x_1, \ldots, x_n$.

For a binomial variable $X$ with $n$ trials,

\[ P\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k} \]

Cumulative distribution:

\[ P\{X \leq k\} = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \]

Expectation and variance next time.
Quiz: Discrete random variables (what was covered today).

HW:
1. Two fair dice are rolled. $X =$ product of the numbers that come up.
   Compute the probability mass function.

2. Two fair dice are rolled. $Y =$ sum of the numbers that come up.
   Compute $E(Y)$, var $(Y)$. 
3. Suppose $E(X) = 1$, $\text{var}(X) = 5$. Find
   (a) $E((2 + X)^2)$
   (b) $\text{var}(4 + 3X)$.

HW due on Tue 5/26.