This material will not be on Exam 2, it will be on Exam 3.

Continuous Random Variables

"Sample space, \( X : S \rightarrow \mathbb{R} \) random variable

(\text{measurable} : \{ X \leq x \} \text{ is a measurable event for every } x \in \mathbb{R}).

We say that the random variable \( X \) is
(absolutely) continuous if there exist a function
\[ f : \mathbb{R} \to [0, \infty] \]
such that
\[ P \{ X \leq x \} = \int_{-\infty}^{x} f(t) \, dt \]

Cumulative distribution

\[ f \] is called probability density.

\[ 1 = P \{ X < \infty \} = \int_{-\infty}^{\infty} f(t) \, dt \]
\[ P \{ a \leq X \leq b \} = \int_a^b f(t) \, dt \]

Same

\[ \int_1^n = \text{area under the whole curve of } f \]
Example 1a  p. 187

Suppose \( X \) is a continuous random variable whose probability density function is

\[
    f(x) = \begin{cases} 
        C(4x - 2x^2) & 0 < x < 2 \\
        0 & \text{otherwise}
    \end{cases}
\]

(a) Find \( C \).

(b) Find \( P\{X > 1\} \).
Solution: (a) We must have \[ \int_0^2 c(4x - 2x^2) \, dx = 1. \]

\[
C = \frac{1}{\int_0^2 (4x - 2x^2) \, dx}
\]

\[
\int_0^2 (4x - 2x^2) \, dx = \left[ 2x^2 - \frac{2x^3}{3} \right]_0^2 = \frac{8}{3} - \frac{16}{3} = \frac{8}{3}
\]

\[
C = \frac{3}{8}
\]
(b) \[ \frac{3}{8} \int_{0}^{2} (4x - 2x^2) \, dx = \frac{3}{8} \left[ \frac{2x^2}{2} - \frac{2x^3}{3} \right]_{0}^{2} = \frac{3}{8} \left( \frac{8}{2} - \frac{4}{3} \right) = \frac{1}{2}. \]

Example 16: The amount of time in hours that a computer functions before breaking down is a continuous random variable with probability density given by

\[ f(x) = \begin{cases} 
\lambda e^{-x/100} & x \geq 0 \\
0 & x < 0 
\end{cases} \]

What is the probability that
(a) A computer will last between 50 and 150 hours before breaking down.

(b) It will last fewer than 100 hours.

Solution: \[ I = \int_{-\infty}^{\infty} e^{-x/100} \, dx \]

\[ d = \int_{-\infty}^{\infty} e^{-x/100} \, dx \]

\[ \int_{0}^{\infty} e^{-x/100} \, dx = \left[ -100 e^{-x/100} \right]_{0}^{\infty} = 0 + 100 = 100 \]

\[ x = \frac{1}{100} \]
\[ P \{ 50 < X < 150 \} = \frac{1}{100} \int_{50}^{150} e^{-x/100} \, dx = \]

\[ \leq \]

\[ (\text{does not matter}) \quad \frac{1}{100} \left[ -180 e^{-x/100} \right]_{50}^{150} = \]

\[ \leq \]

\[ = -e^{-3} + e^{-1} \approx 0.284. \]

\[ P \{ X < 100 \} = \frac{1}{100} \int_{0}^{100} e^{-x/100} \, dx = \]

\[ = [e^{-x/100}]_{0}^{100} = 1 - e^{-1} \approx 0.633. \]
Example 1c: The lifetime in hours of a radio tube is a random variable with probability density function given by

\[ f(x) = \begin{cases} 
0 & x \leq 100 \\
\frac{100}{x^2} & x > 100 
\end{cases} \]

What is the probability that exactly 2 out of 5 such tubes in a radio set will have to be replaced within the first 150 hours of operation? (Assume
the events of such a replacement are independent. \( E_1, \ldots, E_5 \) \( E_i = \{ \text{the } i^{th} \text{ tube has to be replaced} \} \)

\[
P (E_i) = \int_0^{E_i} f(x) \, dx = \int_0^{150} \frac{100}{x^2} \, dx = 100 \int_0^{150} \frac{1}{x^2} \, dx = 100 \left[ -\frac{1}{x} \right]_0^{150} = 100 \left( -\frac{1}{150} - \left( -\frac{1}{100} \right) \right) = 100 \left( \frac{1}{100} \right) = 1 - \frac{2}{3} = \frac{1}{3}.
\]

Bionomial distribution:
\[ \left( \frac{5}{2} \right) \left( \frac{1}{3} \right)^2 \left( \frac{2}{3} \right)^3 = \frac{80}{243} \]

If \( F(a) \) is the cumulative distribution,

then the density is the derivative:

\[ f(a) = \frac{d}{da} F(a) \]

(fundamental theorem of calculus).

Example: A function applied to a continuous
needed in the theory of statistics

random variable $X$: $g(x)$

(Assume $g'(x) > 0$ for all $x$.)

If $f$ is the probability density of $X$, what is the probability density of $g(X)$?

Solution: $P\{X \leq x\} = \int_{-\infty}^{x} f(t) \, dt$

$P\{g(X) \leq y\} = P\{X \leq g^{-1}(y)\} = \int_{g^{-1}(y)}^{\infty} f(t) \, dt$
\[ = \int_{-\infty}^{\tilde{y}(y)} f(t) \, dt = \int_{-\infty}^{\tilde{y}} \frac{f(g^{-1}(s))}{g'(g^{-1}(s))} \, ds \]

Substitute: 
\[ s = g(t), \quad t = g^{-1}(s) \]

\[ t = g^{-1}(y), \quad ds = g'(t) \, dt = g'(g^{-1}(s)) \, dt \]

\[ s = \tilde{y}, \quad dt = \frac{1}{g'(g^{-1}(s))} \, ds \]

Answer: The probability density of \( g(X) \) as:
\[ \frac{f(g^{-1}(x))}{g'(g^{-1}(x))} \]
Example 1d: If $X$ has probability density $f(x)$, what is the probability density of $2X$?

Solution: $g(x) = 2x$, $g^{-1}(x) = \frac{x}{2}$

$g'(x) = 2$, $g'(g^{-1}(x)) = 2$

Answer: $\frac{f\left(\frac{x}{2}\right)}{2}$.

Non-linear example: Let $X$ be a continuous random
Random variable with probability density

\[ f(t) = \begin{cases} 
2t & \text{if } 0 \leq t \leq 1 \\
0 & \text{else.}
\end{cases} \]

What is the probability density of \( X^2 \)?

\[ q(x) = x^2 \]

\[ \frac{f(g^{-1}(x))}{g'(g^{-1}(x))} \]

\[ 0 \leq x \leq 1 \]

Increasing mapping

\[ [0,1) \rightarrow [0,1] \]

\[ 2 \sqrt{x} = 1 \]

\[ \frac{1}{2 \sqrt{x}} \]
Answer: The density of $X^2 = 1$ for $0 \leq x \leq 1$

0 else.

From first principles (same as general formula, but working with concrete functions):

\[ P(X^2 \leq s) \text{ if and only if } X \leq t \]

\[ 0 \leq t \leq 1 \]

\[ 0 \leq s \leq 1 \]

\[ f(t) dt = h(s) ds \]

\[ 2t dt = ? ds \]

transforming densities
\[ ds = 2t \, dt \]

\[ 2t \, dt = 2t \, dt = ds \]

\[ h(s) = 1 \]

**Example:** Suppose \( X \) has probability density

\[ f(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{else} \end{cases} \]

Find the probability density of \( e^X \).
\[ f(t) = 1 \quad s = e^t \quad 0 \leq t < 1 \]

\[ 1 \leq s \leq e \quad t = \ln s \]

\[ f(t) \, dt = h(s) \, ds \]

\[ f(t) \, dt = 1 \cdot dt = \frac{1}{s} \, ds \]

\[ dt = \frac{1}{s} \, ds \]

\[ h(s) = \begin{cases} \frac{1}{s} & \text{for } 1 \leq s \leq e \\ 0 & \text{else} \end{cases} \]

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Expectation and variance X continuous random variable
\[ E(X) = \int_{-\infty}^{\infty} x f(x) \, dx \quad \text{where } f(x) \text{ is the probability density.} \]

\[ \text{if it exists!} \]

\[ E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx \]

\[ \text{var}(X) = E(X^2) - (E(X))^2 \]

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**Example 2a p. 190**

Suppose \( X \) has probability density function \( f(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases} \).

Compute \( E(X), \text{ var}(X) \).
Solution: \[ E(X) = \int_0^1 x f(x) \, dx = \int_0^1 2x^2 \, dx = \left[ \frac{2}{3} x^3 \right]_0^1 = \frac{2}{3} \]

\[ E(X^2) = \int_0^1 x^2 f(x) \, dx = \int_0^1 2x^3 \, dx = \left[ \frac{1}{2} x^4 \right]_0^1 = \frac{1}{2} \]

\[ \text{var}(X) = E(X^2) - (E(X))^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18} \]

Example 26:
\[ f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases} \]

Find \( E(e^X) \).

Solution:

\[
\int_{0}^{1} e^x f(x) \, dx = \int_{0}^{1} e^x \, dx = e^1 \bigg|_0^1 = e - 1.
\]

Example 2c p. 192: A stick of length 1 is pulled at a point \( U \) that is uniformly distributed over \((0, 1)\) (probability density is constant). Determine the expected length of the piece which contains the point \( U \).
for a given $0 \leq p \leq 1$.

Solution: $L_p(U) =$ length of the piece containing $p$.

$$L_p(U) = \begin{cases} 1-u & \text{if } U < p \\ u & \text{if } U > p \\ 0 & \text{otherwise} \end{cases}$$

$$E(L_p(U)) = \int L_p(u) \, du =$$

$$= \int_0^1 (1-u) \, du + \int_1^p u \, du =$$

$$= \frac{1}{2}(1 - 2p + p^2)$$
\[
\int_{0}^{p} \left[ -\frac{1}{2}(1-u)^2 \right] \, du + \int_{p}^{1} \left[ \frac{1}{2} u^2 \right] \, du = \frac{1}{2} - \frac{1}{2} (-p)^2 + \frac{1}{2} - \frac{1}{2} p^2 = \frac{1}{2} - p^2 + p = \frac{1}{2} + p(1-p).
\]

Example 2d p. 193

- c minutes early ... cost cs
- k minutes late ... cost ks

Travel time is random, probability density f. When should you depart to minimise expected cost?

Solution: X = Travel time.
\[ \text{cost: } C_t(X) = \begin{cases} c(t - X) & \text{if } X \leq t \\ k(X - t) & \text{if } X > t. \end{cases} \]

\[ E(C_t(X)) = \int_0^\infty C_t(x) f(x) \, dx = \int_0^t c(t - x) f(x) \, dx + \int_t^\infty k(x - t) f(x) \, dx = c t \int_0^t f(x) \, dx - c \int_0^\infty x f(x) \, dx + k \int_0^t x f(x) \, dx - k t \int_0^t f(x) \, dx. \]
\[
\frac{d}{dt} E(C_t(X)) = c t f(t) + c F(t) - c t f(t) + k t f(t) - k t f(t) - k [1 - F(t)] = (k + c) F(t) - k.
\]

\[(k + c) F(t) - k = 0\]

\[F(t) = \frac{k}{k + c}\]

\[t > F^{-1}\left(\frac{k}{k + c}\right)\]

cumulative distribution.

HW: let \(X\) be a continuous random variable with probability
density function
\[ f(x) = \begin{cases} 
\sqrt{x} & 0 \leq x \leq 1 \\
0 & \text{else}.
\end{cases} \]

1. Find C

2. Find \( P \left\{ \frac{1}{2} \leq X \leq 1 \right\} \)

3. Find \( E(X), \ \text{var} (X) \)

4. Find the probability density of \( X^3 \).