Gaussian variables

Gaussian integral: \( \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{2\pi} \).

Standard Gaussian variable \( X \) has probability density function

\[
\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \mathbb{E}(X) = 0 \quad \text{var}(X) = 1
\]
Gaussian variable with expected value $\mu$ and variance $\sigma^2$ has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Changing the variance:

$$u = ax \quad du = adx$$

$$\int_{-\infty}^{\infty} e^{-\frac{u^2}{2\sigma^2}} du = \frac{1}{a} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = \frac{\sqrt{2\pi}}{a}$$

We can make a random variable $X$ with probability density function

$$f_X(x) = \frac{a}{\sqrt{2\pi}} e^{-\frac{a^2 x^2 / 2}{2\pi}}$$
\[ \text{var}(X) = E(X^2) = \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} \, dx = \quad u = ax \\
\quad du = adx \]

\[ \frac{1}{\sqrt{2\pi} \cdot a} \int_{-\infty}^{\infty} u^2 e^{-\frac{u^2}{2a^2}} \, du = \frac{1}{a^2} \]

If I want a Gaussian variable \( X \) with
\[ E(X) = 6 \quad \text{var}(X) = a^2 \quad (\sigma(X) = a) \]

Probability density function:
\[
\frac{1}{a\sqrt{2\pi}} e^{-\frac{(x-6)^2}{2a^2}}
\]

The general Gaussian (normal) variable.
Looking ahead, suppose we have two random variables \( X, Y \) on the same sample space \( S \):

\[
X: S \to \mathbb{R}
\]

\[
Y: S \to \mathbb{R}
\]

(jointly distributed). We say that \( X, Y \) are independent random variables if for any measurable subsets \( B \subseteq \mathbb{R}, \ c \subseteq \mathbb{R} \)

\[
P \{ X \in B, Y \in C \} = P \{ X \in B \} \cdot P \{ Y \in C \}.
\]
This is the same thing as: \( X \in \mathcal{B}, \ Y \in \mathcal{C} \) are independent events.

**Example:** Throwing two dice, \( X = \) number on die 1, \( Y = \) number on die 2.

**Theorem:** If \( X, Y \) are independent random variables on a sample space \( \mathcal{S} \) then

\[
E(XY) = E(X)E(Y).
\]

For any two random variables \( X, Y \) on a sample space \( \mathcal{S} \),

\[
E(X + Y) = E(X) + E(Y).
\]
If $X, Y$ are independent

$$E((X+Y)^2) = E(X^2 + 2XY + Y^2) = E(X^2) + E(2XY) + E(Y^2)$$

Subtract

$$= E(X^2) + 2E(X)E(Y) + E(Y^2)$$

$$(E(X+Y))^2 = (E(X))^2 + 2E(X)E(Y) + (E(Y))^2$$

$$\text{var} (X+Y) = \text{var}(X) + \text{var}(Y) \leq \text{provided } X, Y \text{ are independent.}$$

Theorem: The sum of two independent Gaussian variables $X, Y$ on a sample space $S$ is Gaussian. To be proved.
Example: The \textit{binomial distribution} probability $p$ and $n$ trials is the sum of $n$ \textit{independent random variables} ($\sim$ \textit{Bernoulli} with probability $p$).

The claim is, in large $n$, the \textit{binomial variable} is \textit{approximately a Gaussian variable}.

\[ X_1 + \cdots + X_n + X_{n+1} + \cdots + X_{n+m} \]

\sim \text{Gaussian} \quad \sim \text{Gaussian} \quad \sim \text{Gaussian} \quad \sim \text{Gaussian} \quad \sim \text{Gaussian} \quad \sim \text{Gaussian} \quad \sim \text{Gaussian} \]
Proof that a sum of two independent Gaussians $X, Y$ is a Gaussian.

Probability density function for $X$:

$$f(x) = \frac{1}{\sqrt{2\pi \sigma_x}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}$$

for $Y$:

$$f(y) = \frac{1}{\sqrt{2\pi \sigma_y}} e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}$$

If the probability density function for $X$ is $f(x)$, $Y$ is $g(y)$, and they are independent then the density

$$f(x, y) = f(x) g(y)$$
Function for $x + y$ is

$$h(x) = \int_{-\infty}^{\infty} f(x) g(z-x) \, dx$$

For Gaussian $X, Y$

$$\text{Constant} \int_{-\infty}^{\infty} e^{-\frac{(x-b)^2}{2} + c(z-x-d)^2}{/2} \, dx = h(x)$$

Idea: bring the exponent to perfect square by adding a linear function of $x$ squared. $\Rightarrow$ no $x$ allowed

$$\frac{(x+z)}{2} + \text{constant}$$
\[ a^2(x-b)^2 + c^2(t-x-d)^2 = C x^2 + (c^2 \alpha + \beta) x + \frac{\gamma x^2 + \delta x + \epsilon}{C} \]

\[ = C \left( x^2 + \alpha x + \frac{\beta}{C} x + \frac{\gamma x^2 + \delta x + \epsilon}{C} \right) \]

\[ = C \left( x + \frac{\alpha x + \beta}{2C} \right)^2 + \text{quadratic in } x \]

\[ \text{de Gauss van integral } \text{ drops out } \]

\[ \text{it is a Gaussian distribution!} \]

\[ \text{How to normalise the homomorad to get a Gaussian?} \]
Take a limit $n \to \infty$, perform linear transformation to keep a fixed expected value and variance.

For a binomial with probability $p$ and $n$ independent Bernoulli trials:

$E = np$

$\text{var} = np(1-p)$

NEXT TIME

HW (1) 5.21 $p \cdot 225$ (table on p. 201 ?)
(2) 5.24 $p \cdot 226$