The $X^2$ distribution with $k$ degrees of freedom

= sum of $k$ independent random variables

which are squares of standard Gaussian variables.

The task: Write down the probability density function.

$k = 1$: $X = Y^2$ where $Y$ is a standard Gaussian.

Probability density function of $Y$

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$
Probability density for \( X \) is related by a change of variable

\[
x = y^2 \quad \text{dx} = 2y \, dy
\]

\[
y = \sqrt{|x|} \quad \text{dx} = 2\sqrt{x} \, dy \quad (x \geq 0)
\]

\[
dy = \frac{dx}{2\sqrt{x}}
\]

\[
\frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy = \frac{1}{\sqrt{2\pi}} e^{-x/2} \cdot \frac{dx}{2\sqrt{x}}
\]

But we have two values of \( y \) to each value of \( x \) (except \( x = 0 \) has measure 0).

So we must multiply by 2:

\[
\frac{1}{\sqrt{2\pi}} e^{-x/2} = g(x), \quad x \geq 0
\]

\[
g(x) = 0 \quad x < 0.
\]
$g(x) = \text{probability density function in } X^2$
with one degree of freedom.

For general $k$: We have a sum of $k$

independent variables each of which has
the distribution $X^2$ with $k = 1$.

$X_1 + \cdots + X_k = x$

$\int \cdots \int g(x_1) \cdots g(x_k) = \int \cdots \int (x_1 \cdots x_k)^{-(k-1)/2} e^{-x/2} \text{ d}x_1 \cdots \text{ d}x_k$

$X_1 + \cdots + X_k = x$

how does this depend on $x$?
\[ u_1 = \frac{x_1}{x}, \ldots, u_k = \frac{x_k}{x}, \quad u_1 + \cdots + u_k = 1 \]

\[ du_1 = \frac{dx_1}{x}, \ldots, du_k = \frac{dx_k}{x} \]

\[ dx_1 = x \, du_1, \ldots, dx_k = x \, du_k \}

we only need \((k-1)\) of them.

So the substitution gives:

\[ \frac{1}{x^{k-1}} \int \ldots \int (u_1 - u_k)^{-1/2} \cdot x^{-k/2} \]

\[ u_1 + \cdots + u_k = 1 \]

\[ x_1 = x \, u_1 \]

\[ = \frac{(k-1)-k/2}{x} \int \ldots \int (u_1 - u_k)^{-1/2} \]

\[ u_1 + \cdots + u_k = 1 \]

\[ \leq \text{does not depend on } x. \]
The $X^2$ distribution with $k$ degrees of freedom is

\[ C \frac{k-1}{2} x^{k-1} e^{-x/2} \]

\[ 0 \quad x < 0 \quad x > 0 \]

What is the constant $C$?

\[ C = \frac{1}{\int_0^\infty x^{k-1/2} e^{-x/2} \, dx} \]

The standard notation: The Gamma ($\Gamma(-)$) function.

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \quad z > 0. \]
The gamma function is
not an elementary
function. And closing
integration by parts,

\[ u = t - 1 \]

\[ v = e^t \]

\[ \int e^t \, dt = e^t + C \]

(Notes: complex analysis
complex number \( \mu \).

\( \mu = 0, 1, 2, \ldots \))

\( (z-2)(z-3)(z-4)(1-\alpha) = \)

\( \prod_{j=1}^{\infty} (z-\gamma_j) \]

The convergence of
the Taylor series
except for \( z = 0, 1, 2, \ldots \).
\[ \Gamma(0) = \int_0^\infty e^{-t} \, dt = \left. -e^{-t} \right|_0^\infty = 1 \]

If \( n \) is a positive integer,

\[ \Gamma(n) = (n-1)! \]

Now back to calculating the constant \( C \)

as above:

\[ \frac{1}{C} = \int_0^\infty x^{\frac{k}{2} - 1} e^{-x/2} \, dx \quad \text{Put} \quad t = \frac{x}{2} \]

\[ = 2^{\frac{k}{2}} \int_0^\infty t^{\frac{k}{2} - 1} e^{-t} \, dt = 2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right) \]

\[ x = 2t \quad dx = 2 \, dt \]
The $X^2$ distribution has probability density function

$$\begin{align*}
\frac{1}{\frac{1}{2} - 1 - x/2} & \quad x \geq 0 \\
\frac{1}{2^{\frac{1}{2}}} e^{-x/2} & \quad x < 0.
\end{align*}$$

When $k = 1$:

$$\frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}} e^{-x/2}$$

$$2^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) = \sqrt{2\pi}$$

Derive a formula for $\Gamma\left(k + \frac{1}{2}\right), \ k \in \mathbb{Z}$.
2. Compute the hazard rate function for the $X^2$ distribution with $k$ degrees of freedom.