1. Prove that the abelian group $\mathbb{Z}/n^2\mathbb{Z}$ is never isomorphic to $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ for any integer $n > 1$.

**Solution:** $\mathbb{Z}/n \oplus \mathbb{Z}/n$, which has $n^2$ elements, is not cyclic because each of its elements $a$ satisfies $na = 0$ and hence the subgroup generated by $a$ has at most $n$ elements. □
2. Let \( f : A \to \mathbb{Z} \) be a homomorphism of abelian group which is onto. Prove that then there exists an abelian group \( B \) and an isomorphism \( \phi : A \to B \oplus \mathbb{Z} \) such that the following diagram commutes

\[
\begin{array}{ccc}
A & \xrightarrow{f} & \mathbb{Z} \\
\downarrow{\phi} & & \downarrow{\pi} \\
B \oplus \mathbb{Z} & \nearrow{} &
\end{array}
\]

where \( \pi \) is the projection to the second factor.

**Solution:** Put \( B = Ker(f) \). Choose \( a \in A \) such that \( f(a) = 1 \), and define \( \phi(b) = (b - f(b)a, f(a)) \). Then \( \phi(a) = (?, 1), \phi(b) = (b, 0) \) for \( b \in B \), and hence \( \phi \) is onto. If \( \phi(x) = 0 \), then by definition, \( f(x) = 0 \) so \( x \in Ker(f) = B \), so \( \phi(x) = (x, 0) \) and hence \( x = 0 \) and \( \phi \) is also injective. \( \square \)
3. Describe all the left ideals in the ring $M_2(\mathbb{R})$ of $2 \times 2$ matrices over $\mathbb{R}$.

   Solution: (Rohini’s idea) All the left ideals of $M_n(\mathbb{R})$, identified with the ring of linear maps $\mathbb{R}^n \to \mathbb{R}^n$, for any $n$, are the sets $I_V$ of all matrices annihilating a given vector subspace $V \subseteq \mathbb{R}^n$ (i.e. there is one left ideal for each $V$). Clearly, these are ideals. On the other hand, denote for $A \in M_n(\mathbb{R})$ by $V(A)$ the solution space $\{x \in \mathbb{R}^n \mid Ax = 0\}$. Let $I \in M_n(\mathbb{R})$ be a left ideal and let $A \in I$ be such that the solution space has the least dimension $k$. Then WLOG, $V(A)$ is spanned by the last $k$ coordinate vectors in $\mathbb{R}^n$, hence using row operations, the diagonal matrix $D_{n-k}$ with first $n - k$ diagonal entries 1 and other diagonal entries 0 satisfies $D_{n-k} \in I$. Hence, any matrix with the last $k$ columns equal to 0 is in $I$. Hence, no matrix whose last $k$ columns are not all zero is in $I$ (since otherwise we may alter the first $n - k$ columns arbitrarily and thus produce a matrix with a solution space of lower dimension). $\square$
4. Construct a commutative ring $R$ containing $\mathbb{R}[x]$ as a subring such that $x, x + 2 \in R^\times$ but $x + 1 \notin R^\times$. [It is OK to construct an injective homomorphism $\mathbb{R}[x] \to S$ and treat it as an inclusion.]

**Solution:** Clearly, letting $D$ be the set of all polynomials of the form $x^k(x + 2)^\ell$, we want to put $R = D^{-1}\mathbb{R}[x]$. The universal ring homomorphism $\mathbb{R}[x] \to R$ is an inclusion since $D$ does not contain 0 or any zero divisors. Why is $x + 1$ not a unit of $R$? Otherwise,

$$(x + 1)p(x)/(x^k(x + 2)^\ell) = 1$$

in $R$ and hence in the field of rational functions $\mathbb{R}(x)$ by universality. This implies

$$(x + 1)p(x) = x^k(x + 2)^\ell \in \mathbb{R}[x],$$

but considering roots, that is impossible. □
5. Let $R, S$ be commutative rings. Prove that prime ideals in the product ring $R \times S$ are precisely subsets of the form

$$R \times q = \{(x, y) \in R \times S \mid x \in R, y \in q\}$$

for a prime ideal $q$ of $S$ and

$$p \times S = \{(x, y) \in R \times S \mid x \in p, y \in R\}$$

for a prime ideal $p$ of $R$.

**Solution:** One readily verifies by definition that $R \times q$, $p \times S$ for $p \subset R$ and $q \subset S$ prime are prime ideals of $R \times S$. To prove the converse, let $I$ be a prime ideal of $R \times S$. Then $(0, 1) \cdot (1, 0) = (0, 0) \in I$, so $(0, 1) \in I$ or $(1, 0) \in I$. WLOG $(0, 1) \in I$. Letting $p$ be the image of $I$ under the projection $R \times S \to R$, we already see that $I = p \times S$. To see that $p$ is prime, suppose $x, y \notin p$, $xy \notin p$. Then $(x, 0), (y, 0) \notin I$, $(x, 0) \cdot (y, 0) = (xy, 0) \in I$ - a contradiction with $I$ being prime. \qed