Theorem: Let $R$ be a commutative ring and let $S \subseteq R$ closed under multiplication ($x, y \in S \Rightarrow xy \in S$) (and such that $1 \in S$). Then there exists a commutative ring $S^{-1}R$ and a ring homomorphism 

$$\pi : R \rightarrow S^{-1}R$$

such that for every homomorphism of rings $f : R \rightarrow Q$ where $f(S) \subseteq Q^X$ there exist a unique homomorphism of rings $\overline{f} : S^{-1}R \rightarrow Q$ such that the following diagram commutes:
Proof: We would like to say that the element of $S^{-1}R$ are of the form $\frac{r}{s}$ where $r \in R$, $s \in S$.

Technically, we start with

$$\{(r,s) \mid r \in R, s \in S\} = R \times S$$

We impose an equivalence relation $\sim$ on $R \times S$ such that

$$\frac{r}{s} \sim \frac{r'}{s'} \iff \frac{r}{s} = \frac{r'}{s'} \iff rs' = s'rs$$
\[(r, s) \sim (r', s') \quad \text{if and only if} \quad \exists u \in S \quad \text{such that} \quad rs'u = s'r'u.\]

Good guess, but not quite enough.

We let \(S'\) be the set of equivalence classes of \(\sim\).

\[
\begin{align*}
(r_1, s_1) \cdot (r_2, s_2) &= (r_1 r_2, s_1 s_2) \\
(r_1, s_1) \sim (r_1', s_1') \\
(r_1 r_2, s_1 s_2) \sim (r_1', r_2', s_1', s_1')
\end{align*}
\]

\[
\begin{align*}
rs'u &= r's'u \\
rs'v &= r's'v \\
rs's'un &= rs's'v \\
rs's'vn &= rs's'v \\
rs's'vn &= rs's'v \\
rs's'vn &= rs's'v \\
rs's'vn &= rs's'v \\
rs's'vn &= rs's'v
\end{align*}
\]
\[(r_1, s_1) + (r_2, s_2) := (r_1 s_2 + r_2 s_1, s_1 s_2) \quad \text{Assoc. easy}\]

\[(r_1, s) \sim (r_1', s') \quad r_1 s' u = r_1' s u\]

\[(r_1 s_2 + r_2 s_1, s_1 s_2) \sim (r_1' s_2 + r_2 s_1', s_1' s_2)\]

\[\exists v \in S\]

Choose \( v = u \).

\[1 = (1, 1) \quad (= (s, s) \quad \text{for any } s \in S)\]

\[(\text{if } S = \emptyset \quad S \cdot R = R)\]
\[ 0 = (0, 1) \]

\[ (r_1, s_1) + (0, 1) = (r_1 + s \cdot 0, s \cdot 1) = (r_1, s) \]

\[ -(r_1, s) = (-r_1, s) \]

\[ (r_1, s) + (-r_1, s) = (r_1 - r_1, s + s) = (0, 2s) = (0, s^2) = 0 \]

\[ \left( (r_1, s_1) - (r_3, s_3) \right)^2 = (r_1, s_1) - (r_3, s_3) = (r_1, s_1) (r_3, s_3) + (r_3, s_3) (r_1, s_1) \]

\[ \left( r_1 s_2 - r_2 s_1 + r_2 s_1 s_2 s_3 + r_3 s_1 s_2 s_3 \right) \]

\[ \left( r_1 s_2 s_3 - r_2 s_1 s_2 s_3 \right) \]

\[ s_1 s_2 s_3^2 \]
\[r_1 s_1 s_2 s_3 + r_2 s_1 r_3 s_2 s_3 = r_1 s_1 s_2 s_3 + r_2 s_1 s_2 s_3 \]

\[s_1 s_2 s_3.\]

**Universality:**

\[\pi : R \rightarrow s^{-1} R\]

\[r \mapsto (r, 1) = \frac{r}{1}\]

Call \((r, s)\)

\[
\frac{r}{s}
\]

If \(f : R \rightarrow \mathbb{Q}\) satisfies \(f(s) \in \mathbb{Q} \times \mathbb{Q}\)

we can define

\[f\left((r, s)\right) = \frac{f(r)}{f(s)}\]

That is a homomorphism. \(\circ\) is forced

\[\left((r, s) \cdot (s, 1) = (r, 1) \bigg| \frac{r}{s} \cdot s = r\right)\]
\[
\bar{f}(r,s) \cdot f(s) = f(r) \quad \forall \in \mathbb{Q}^x
\]

\[
\bar{f}(r,s) = f(r)^{-1} f(s). \quad \square
\]

**Proposition:** If \( S \neq 0 \) and \( S \) contains any \( 0 \) divisors then \( \Pi: \mathbb{R} \rightarrow S^{-1} \mathbb{R} \) is injective.

**Proof:** \( \Pi : r \rightarrow (r,1) \). Injectivity means:
\((r, s) \sim (r', s)\) \quad \text{then} \quad r = r' \in R.

\[ ru = r'u , \quad u \in S \]

\[(r-r')u = 0 \quad u \in S \quad u \neq 0\]

But then \(r-r' = 0\) because \(u\) is not a \(0\)-divisor.

Some important cases of localization:

1. If \(p\) is a prime ideal in \(R\) then

\[ R_p = S^{-1}R \quad \text{where} \quad S = R \setminus p. \]

\(R_p\) is closed under \(\cdot\) because \(p\) is a prime ideal.
2) When $R$ is an integral domain, then (if and only if)

$(0) = \{0\}^3$ is a prime ideal in $R$.

$$R_0 = S^{-1}R \quad S = R \setminus \{0\}$$

is a field.

$R \subseteq \bigcirc S^{-1}R$ This is called the fraction field of $R$.

$\mathbb{Q}$ is the fraction field of $\mathbb{Z}$

If $k$ is a field,

$k(x) := \text{the fraction field of } k[x].$

\text{rational functions}
Quiz on Wednesday

HW
① 3 on p. 264
② 6 on p. 265