Quit solution continued:
\[ \mathbb{Q}[x] \]
\[ \mathbb{Q} \text{ is divisible by } (x^2 - 7). \]

leading coefficient
\[ v = 1 \text{ (monic polynomial)} \]

can perform long division,
are to remainders, so I can assume \( \deg p_1, \deg q < 2 \)

Then \( p_1, q \) both have to be of degree 1.

\[ (x^2 - 7) \mid \mathbb{Q}[x] \Rightarrow \mathbb{Q}[x] \]

roots \( \pm \sqrt{7} \) \( \Rightarrow \) \( p \) and \( q \) must have root \( \sqrt{7} \) contradiction
\[ p(x) = a(x)(x^2 - 7) + r_1(x) \]
\[ \deg < 2 \]
\[ q(x) = b(x)(x^2 - 7) + r_1(x) \]

\[ (x^2 - 7) \mid p(x) \mid q(x) \Rightarrow (x^2 - 7) \mid p_1(x) \mid r_1(x). \]

For exam, please review base matrix algebra over a field (1 problem about matrix rings) - Gauss elimination - rank of row or column operations - solving linear equations - inverse matrix
Slightly more geometry: \( R \) commutative ring

\[
\text{Spec } R = \{ \text{all prime ideals of } R \}
\]

Zariski topology: closed set: \( Z(I) \), \( I \subseteq R \)

\[
Z(I) = \{ \text{prime ideals } p \mid I \subseteq p \}
\]

Open sets are complements of closed sets.

Localization enables us to define the "ring of functions" on \( \text{Spec } R \setminus Z(I) \).

Easy if \( I = (f) \): Answer \( S^{-1} R \), \( S := S_f \)

\[
S = \{ 1, f, f^2, f^3, \ldots \}
\]
\[ Z(f) = \text{"set of zeroes of the function } f \text{"} \]

"We should be allowed to divide by a function when we are away from its set of zeroes."

In the case of a general \( I \), we want "all the functions which are regular on \( \text{Spec } R \setminus Z(f) \) for all \( f \in I \)."

\[
\begin{align*}
\{ (a_f) \mid & \quad \pi_1(a_f) = \pi_2(a_f) \} \\
\longrightarrow & \\
\prod_{f \in I} S_f & \cong R
\end{align*}
\]

regular functions on \( f \) by definition

\[
\begin{align*}
\prod_{(f,g) \in I \times I} S_{fg} \cdot R
\end{align*}
\]
(a_f | f ∈ I) → (b_{fg} | (f, g) ∈ I × I)

b_{fg} = a_f

-> ...

b_{fg} = a_g

Divisibility Theory - (the other side of commutative rings)

Modelling concepts of number theory within commutative rings

Chinese remainder theorem
**Intro:** Suppose $A, B$ are ideals in $R$ (commutative).

$A + B = \{ a + b \mid a \in A, b \in B \}$.

**Claim:** $A + B$ is an ideal. □

**Theorem:** Let $A_1, \ldots, A_k$ be ideals in $R$ (commutative).

Define $\varphi : R \longrightarrow R/A_1 \times \cdots \times R/A_k$

$$\varphi : r \longmapsto (r + A_1, \ldots, r + A_k)$$

is a ring homomorphism with kernel $A_1 \cap \cdots \cap A_k$.

If, in addition, $A_i + A_j = R$ for all $i \neq j$ then $\varphi$ is also onto.

Also, $A_1 \cap \cdots \cap A_k = A_1 \cdot \cdots \cdot A_k$.
(By homomorphism theorem, then,)

\[ \overline{\varphi} : R/A_1 \times \cdots \times A_k \rightarrow R/A_1 \times \cdots \times R/A_k. \]

(Note: Generality !!!)


\[ \ker \varphi = \{ x \in R \mid x + A_1 = 0 \in R/A_1, \ x + A_2 = 0 \in R/A_2 \} \]

\[ = \{ x \in R \mid x \in A_1, \ x \in A_2 \} = A_1 \cap A_2. \]

Suppose now $A_1 + A_2 = R$. Then $\forall x \in A_1, \ y \in A_2$

\[ x + y = 1. \quad x + A_1 \leftarrow x + A_2 \]

Claim: $\varphi(x) = (0, 1)$ \[ \varphi(y) = (1, 0) \] \[ x + A_2 = 1 - y + A_2 = (+ A_2 \ (y \in A_2)) \]
\[ \phi(sx + ry) = (n,s) \quad \text{for any } n, r \in \mathbb{R} \]

**Induction Step:** Suppose it is true with \( k \) replaced by \( k-1 \). Homomorphism and kernel follow.

\[ \mathbb{R} \longrightarrow \mathbb{R}/(A_1 \cap \cdots \cap A_{k-1}) \times \mathbb{R}/A_k. \]

\[ \downarrow \]

\[ \mathbb{R}/A_1 \times \cdots \times \mathbb{R}/A_{k-1} \times \mathbb{R}/A_k. \]

(Proven directly!)

**Non-trivial part:**

\[ \phi(k = 2) : \text{Showing} \]

\[ A_1 \cap A_2 = A_1 A_2. \]

We always have \( A_1 A_2 \subseteq A_1 \cap A_2 \), equality follows.

\[ c = cx + cy \quad (1 = x + y) \]
But the onto part also follows if we know
\[ A_1 \cdot \ldots \cdot A_{k-1} + A_k = R. \]

We know \( A_i + A_k = R \) \( \forall i \in \{1, \ldots, k-1\} \).

\[ x_i + y_i = 1 \quad x_i \in A_i, \ y_i \in A_k. \]

\[ \ell = (x_1 + y_1) \cdot \ldots \cdot (x_{k-1} + y_{k-1}) \in x_1 \cdot \ldots \cdot x_{k-1} + A_k. \]

\[ \ell \in A_1 \cdot \ldots \cdot A_{k-1} \]
\[
\mathbb{R} \rightarrow \mathbb{R}/A_1 \cap \cdots \cap A_k \times \mathbb{R}/A_k
\]
\[
\downarrow
\]
\[
\mathbb{R}/A_1 \times \cdots \times \mathbb{R}/A_{k-1} \times \mathbb{R}/A_k
\]

HW:
1. p. 267, Problem 1
2. p. 268, Problem 6
3. p. 268, Problem 7

Homework 237 Test