Quit: \( c \)  
Yes: \( a + b \sqrt{-13} \)  
\( a, b \) even \( \in 0 + 1 \)  
\( 2, 2 \sqrt{-13} \)  
\( a, b \) odd \( \in 0 + 1 \)  
\( 1 + \sqrt{-13} + x \cdot 2 + y \cdot 2 \sqrt{-13} \)  
\( x, y \in \mathbb{Z} \)  
\( a \) even, may assume  
\( b \) odd \( a = 0 \)  
\( b = 1 \)  
\( \sqrt{-13} \in 1 + 1 \)  
\( (1 + \sqrt{-13}) \neq 1 (-2) \)
We also need to prove \( I \neq 1 \)

\[
(a + b\sqrt{13}) (1 + \sqrt{13}) = a - 13b + (a + b)\sqrt{13}
\]

\( a - 13b, a + b \) have the same parity

\[
2 \mid 2a - 13b
\]

Quotient \( \mathbb{Z}/I = \mathbb{Z}/2\mathbb{Z} \) field \( \Rightarrow \) ideal is maximal \( \Rightarrow \) prime

(2) Not a UFD because this ideal is not principal.

\[
N(a + b\sqrt{13}) = a^2 + 13b^2
\]

\[
N(1 + \sqrt{13}) = 14 \quad N(2) = 4
\]

\[
(1 + \sqrt{13}, 2) = (a + b\sqrt{13}) \quad a^2 + 13b^2 \mid 4, 14
\]

2, 1
\[ a^2 + 11b^2 = 1 \quad a = \pm 1, \quad b = 0 \quad \text{unit impossible} \]
\[ a^2 + 13b^2 = 1 \quad \text{has no solution} \]
\[ \Rightarrow b = 0. \]

back to leaves: Euclidean domains.

An integral domain \( R \) is a Euclidean domain if there is a function \( N : R \to \mathbb{N}_0 \), \( N(0) = 0 \)

(such a function is called a norm) \( 0, 1, 2, \ldots \)

such that for \( a, b \in R \), \( b \neq 0 \) there exist \( q, r \in R \)

with

\[ a = bq + r \quad (r = 0 \text{ or } N(r) < N(b)). \]
Proposition: Every Euclidean domain is a PID.

↑

Lemma: ∀ a, b ∈ R there exist x, y ∈ R such that ax + by | a, ax + by | b (⇒ ax + by = gcd(a, b)).

& R NOETHERIAN!!

Lemma ⇒ PID : (a₁, ..., a_k). By induction,

∃ x_1, ..., x_k ∈ R, a_1 x_1 + ... + a_k x_k | a_1, ..., a_k

(a_{1y_1} + a_{2y_2}) a_1, a_2 replace (a_1, ..., a_k) by

(a_{1y_1} + a_{2y_2}, a_3, ..., a_k).

⇒ (a_1, ..., a_k) = (a_1 x_1 + ... + a_k x_k). □
Proof of the lemma: The Euclidean algorithm.

Set \( a = a_1, \quad b = a_2 \)

\[
a_k = a_{k+1} r_k + a_{k+2} \quad W(q_{k+1}) < N(q_{k+1})
\]

unless \( a_{k+2} = 0 \)

\[
\begin{align*}
\alpha_1 &= a_2 r_1 + a_3 \\
\alpha_2 &= a_3 r_2 + a_4 \\
\alpha_{k-1} &= a_{k-2} r_{k-1} + a_{k+1} \\
\alpha_k &= a_{k-1} r_k \\
\end{align*}
\]

Recursively:
\( \alpha_{k+1} \) is a linear combination of \( \alpha_{k+1}, a_{k+1} \)
Prove that if \( p \in \mathbb{Z} \) prime, \( \sqrt{-D} \in \mathbb{Z} \) \( \cong \mathbb{Z} \) and \( D < 0 \) then \( (p, \sqrt{-D}) \) is not a principal ideal in the ring of quadratic integers with discriminant \( D \).