MODULES (OVER COMMUTATIVE RINGS)

Let \( R \) be a commutative ring. An \( R \)-module is an abelian group \( M \) together with homomorphisms of abelian groups
\[ r : M \rightarrow M \quad \text{for every } r \in R \]
\[ r(m + m') = rm + rm' \]

1. \( 1 : M \rightarrow M \) is the identity
2. \((r+s)m = rm + sm\)
3. \( r(sm) = (rs)m\).
(Note: Same as axioms of a vector space except 
$k$ does not have to be a field).

\[ 0 \cdot m = 0 \quad (0+0) \cdot m = 0 \cdot m + 0 \cdot m \]
\[ (-1) \cdot m = -m \quad -0 \cdot m \]
\[ (-1) \cdot m + 1 \cdot m = (-1+1) \cdot m = 0 \cdot m \]

Example: A $\mathbb{Z}$-module is the same thing as an abelian group.

\[ k \in \mathbb{Z} \quad k > 0 \quad ka = a + \ldots + a \quad (k \text{ times}) \]
\[ 0 \cdot a = 0 \]
\[ k < 0 \quad ka = (-a) + \ldots + (-a) \quad (-k \text{ times}) \]
Example: When $F$ is a field, an $F$-module is the same thing as an $F$-vector space.

Example: Let $\phi : R \to S$ be a homomorphism of commutative rings. Then any $S$-module $M$ becomes an $R$-module by setting

$$r \cdot m = \phi(r) \cdot m$$

for $r \in R$ and $m \in M$.

The theory of $R$-modules is a rehash of the theory of abelian groups.

A homomorphism of modules $f : M \to N$ homomorphism of abelian groups, $f(rm) = rf(m)$. 
Isomorphism = bijective homomorphism = inverse is also a module, same structure.

submodule \[ N \leq R \]
\[ m \in N \text{ same } \iff m \in N. \]

Free module on a set \( S \):
\[ RS = \{ \phi : S \to R \mid \forall F \subseteq S \text{ finite, } \phi(F) = 0 \text{ if } s \notin F \} \]
\[ \sum_{s \in F} \phi(s)s \]
\[
S \rightarrow RS
\]
\[
s \mapsto s = \psi : S \rightarrow R \quad \psi(s) = 1
\]
\[
\psi(t) = 0 \text{ for } t \neq s.
\]

**Universal property:** For every R-module \(M\) and every map \(f : S \rightarrow M\) there exists a unique homomorphism of modules \(F : RS \rightarrow M\) such that \(F(s) = f(s)\) for \(s \in S\).

**Factor modules:** If \(M\) is an R-module and \(N \subseteq M\) a submodule, then the quotient group \(M/N\)
is an $R$-module by

$$l(m + N) = rm + N.$$ \hfill (1)

\text{Equations: } m + N = m' + N \Rightarrow m - m' \in N \leq N.

$$l(rm + N) = rm - rm' \in N \leq r(m - m') \in N.$$ \hfill (2)

\underline{If} $f : M \to N$ is a homomorphism of $R$-modules

\text{Ker} f = \{ m \in M \mid f(m) = 0 \} \leq \text{a submodule of } M.

\text{Im} f = \{ f(m) \mid m \in M \} \leq \text{a submodule of } N.

\text{Coker} f = N / \text{Im} f.$
The homomorphism theorem: Let \( f : M \to N \) be a homomorphism of \( R \)-modules. Then there exist a unique homomorphism of \( R \)-modules \( \bar{f} : M/\ker f \to N \) such that the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow & & \downarrow \\
M/\ker f & \xrightarrow{\bar{f}} & N
\end{array}
\]

Moreover, \( \bar{f} \) is injective.

An exact sequence of \( R \)-modules is a diagram of the following form:
\[ \mathbb{Z}_1 \xrightarrow{f_1} \mathbb{Z}_2 \xrightarrow{f_2} \mathbb{Z}_3 \rightarrow \cdots \rightarrow \mathbb{Z}_n \leq \text{an abelian group} \leq \text{be infinite on either side} \]

such that \( f_i \) are homomorphisms and

\[ \text{Ker } f_i = \text{Im } f_{i-1}, \text{ where applicable.} \]

**Examples:**

1. \( 0 \rightarrow \mathbb{Z} \rightarrow O \)  
   - exact sequence
   - \( \mathbb{Z} \) is an \( \mathbb{Z} \)-module
   - \( \mathbb{Z} \equiv \mathbb{Z} \)
   - \( O \) is also called \( 0 \).

2. \( 0 \rightarrow M \rightarrow N \rightarrow 0 \)  
   - exact
   - \( f \) is an isomorphism
   - \( \text{Im } f = N \)
Note generally, if \( 0 \to M \to N \) exact means \( \text{Ker} f = 0 \) \( \iff \) \( f \) is injective.

\[ N \xrightarrow{f} N \to 0 \] means \( \text{Im} f = N \subseteq \text{Ker} f \) (when \( f = 0 \)) is onto.

\[ 0 \to M \xrightarrow{i} N \xrightarrow{j} \mathcal{P} \to 0 \] \( \xrightarrow{f} \)

A short exact sequence means \( i \) is injective, \( j \) is onto.

\( \text{Ker} j = \text{Im} i \) homomorphism theorem:

\[ N \xrightarrow{i} \mathcal{P} \xrightarrow{j} \mathcal{Q} \]

\( j \) is injective, \( j \) is onto because \( i \) is injective.

\( N/\text{Im} i = N/\text{Ker} j \) is onto.
"\[ P \cong N/M \]"

\[ \bar{\varphi} \text{ is an isomorphism.} \]

**HW**

1. Prove the universal property for the free module.
2. Prove the homomorphism theorem for \( R \)-modules.
3. Consider a long exact sequence

\[ \ldots \rightarrow M_{n} \rightarrow M_{n-1} \rightarrow \ldots \rightarrow (\text{infinite}) \]

Then it breaks up into short exact sequences:
\[ \begin{array}{c}
1_{M+1} \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \\
0 \rightarrow A_n \rightarrow 0 \rightarrow A_{n-1} \rightarrow 0 \\
\end{array} \]