For the matrix

\[
A_p = \begin{pmatrix}
0 & -a_0 \\
1 & 0 \\
\vdots & \ddots \\
1 & -a_{n-1}
\end{pmatrix},
\]

\[
M_{A_p} = F[x] \langle x \mathbf{I} - A \rangle
\]

\[
\Rightarrow \mathbb{F}[x] / \langle p(x) \rangle
\]

\[
(p = p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0)
\]

\[
a_x \in \mathbb{F}
\]

as an \( \mathbb{F}[x] \)-module (an cyclic module)
A canonical form of matrices with respect to similarity. Let $A$ be an $n \times n$ matrix over $\mathbb{F}$. Let $M_A = \mathbb{F}[x]/\langle p_1(x) \rangle \oplus \cdots \oplus \mathbb{F}[x]/\langle k(x) \rangle$ \hspace{1cm} (*)

where $p_1(x), \ldots, k(x)$ are monic and

$p_1(x) \mid \cdots \mid p_k(x)$. \hspace{1cm} (**)

(Note: no $\mathbb{F}[x]$ (or free) summands are possible because $M_A$ as an $\mathbb{F}$-vector space is an $\mathbb{F}$-module $\mathbb{F} \bigoplus \mathbb{F}[x]$)

is finite-dimensional while $\mathbb{F}[x]$ is not.)
Lemma: In (*) and (**),
\[ \deg P_1(x) + \ldots + \deg P_n(x) = m. \]

This implies that

\[ A \text{ is similar to } \begin{pmatrix} A_{p_1} & 0 \\ 0 & A_{p_2} \\ & \ddots & \ddots \\ & & & A_{p_k} \end{pmatrix} = A_{p_1} \otimes \ldots \otimes A_{p_k} \]

The block sum of matrices.

(Note \( M_{A \otimes B} \cong M_A \otimes M_B \).)
A discussion: It is better to think in terms of equivalence of matrices over \( \mathbb{F}[x] \).

\((Ix-A)\) is equivalent to \((Ix-B)\).

\[ \text{(1)} \quad (Ix-A) = UV (Ix-B) \quad U, V \text{ are invertible matrices over } \mathbb{F}[x]. \]

Take the determinant of (1). The determinant is multiplicative (embedded into the field of fractions \( \mathbb{F}(x) \)).

\[ \det U, \det V \in (\mathbb{F}[x])^\times = \mathbb{F}^\times. \]
\[ \det (I_x - A) = \frac{\det U}{\eta} \cdot \frac{\det (I_x - B)}{\eta} \cdot \frac{\det V}{\eta} \]

\[ \in \mathbb{F}^x \]

\[ = \eta \det (I_x - B), \quad \eta \in \mathbb{F}^x. \]

**I Claim that** \( \eta = 1. \)

\[ \det (I_x - A), \quad \det (I_x - B) \]

are monic polynomials.

\[ \therefore \quad \det (I_x - A) = \det (I_x - B). \]

**The characteristic polynomial of** \( A. \)

**Proof of Lemma:**

But we really need to consider

\[ I_x - A = U P V \]

\( U, P, V \) matrices over \( \mathbb{F}[x], \) \( U, V \) invertible.
\[
U, P, V \in M_n(\mathbb{F}[x])
\]
\[
U, V \in \prod_n(\mathbb{F}[x])^X.
\]

Again,
\[
\det(Ix-A) = \frac{\det U \det P \det V}{\xi \in \mathbb{F}^X} \quad \xi \in \mathbb{F}^X
\]
\[
\det P = m \quad \xi \in \mathbb{F}^X
\]
\[
\therefore \deg(\det P) = \deg \det(Ix-A) = m. \quad \square
\]

The generalized Jordan form

is based on the other canonical form for modules. Let A be an \(m \times m\) matrix over \(\mathbb{F}\).
\[ M_A = \bigoplus_{i=1}^{m} \mathbb{F}[x] \left/ (r_i(x)^{q_i(x)}) \right. \]

The \( q_i(x) \) are prime (= irreducible) monic polynomials \( \in \mathbb{F}[x] \).

As above, we have \( q_1 \deg q_1(x) + \ldots + q_m \deg q_m(x) = m \).

Here is another matrix \( J_i(q_i(x)) \) such that

\[ M_i(q_i(x)) \cong \mathbb{F}[x] \left/ (q_i(x)^{j_i}) \right. \]

as an \( \mathbb{F}[x] \) module.
\[ A_q' = \begin{pmatrix} A_q & O^1 \\ O & A_q \\ & \ddots \\ & & A_q \end{pmatrix} \]

\[ = J_q \cdot \varphi(x) \]

Generalized Jordan block.

\[ A_q = \begin{pmatrix} 0 & -a_0 \\ 1 & -a_0 & \ddots & \ddots \\ & \ddots & \ddots & -a_{n-2} \\ & & \ddots & 1 - a_{n-1} \end{pmatrix} \]

\[ x^i I - A_q = \begin{pmatrix} x & a_0 & \ddots & \ddots \\ -1 & \ddots & \ddots & \ddots \\ & \ddots & \ddots & a_{n-2} \\ & & \ddots & 1 - x + a_{n-1} \end{pmatrix} \xrightarrow{\text{upper triangular}} \begin{pmatrix} 0 & \varphi(x) \\ -1 & 0 \\ \vdots & \ddots & \ddots \end{pmatrix} \]
\[ x \mathbb{1}_{\mathcal{M}_1} \sim \sum_j \left( x_j \right) \]

equivalent

\[
\left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array} \right)
\]

\[
\left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array} \right)
\]

\[
\left( \begin{array}{ccc}
q_1 & 0 & 0 \\
0 & q_2 & 0 \\
0 & 0 & q_2 \\
\end{array} \right)
\]

\[
\left( \begin{array}{ccc}
q_1 & 0 & 0 \\
0 & q_2 & 0 \\
0 & 0 & q_2 \\
\end{array} \right)
\]

\[
\left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array} \right)
\]

Recall \( M_A = \bigotimes_{i=1}^n \mathbb{F}[x]/(q_i(x)^n) \) \( q_i(x) \) irreducible.
The generalized Jordan form of $A$ is

$$A \text{ is similar to } \left( \begin{array}{c}
\int_{\alpha_1} (q_1(x)) \\
\vdots \\
\int_{\alpha_m} (q_m(x))
\end{array} \right) = \bigoplus_{j=1}^{m} \int_{\alpha_j} (q_j(x)) \bigoplus \cdots \bigoplus \int_{\alpha_m} (q_m(x)).$$

A field $\mathbb{F}$ is called algebraically closed if for every polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = p(x)$

$$a_i \in \mathbb{F}, \quad n \geq 1$$
there exist a $2 	imes 2$ such that $p(\lambda) = 0.$

(Clearly, equivalent to saying: all irreducible polynomials in $\mathbb{F}[x]$ have degree 1.)

$$J_k (x - \lambda) = \begin{pmatrix} 
\lambda & 1 \\
0 & \ddots & \ddots \\
& \ddots & \ddots & 1 \\
&& 0 & \lambda 
\end{pmatrix}$$

$$A_{x-\lambda} = (\lambda). \text{ In the algebraically closed case, we just have the old Jordan form.}$$
HW

1. #13 p. 500

2. #15 p. 500

3. Prove that if $A, B$ are matrices over a field $F$, if $C \in E$ where $E$ is another field (e.g. $F = \mathbb{R}, E = \mathbb{C}$) then if $A, B$ are similar over $E$, then they are similar over $F$. 
