Main axes theorem for Hermitian forms over \((\mathbb{C}, \bar{\cdot})\).
(special case: symmetric bilinear form over \(\mathbb{R}\)).

In terms of matrices - Adjoint matrix to \(A\): \(A^* = A^\top\).

Hermitian congruence of matrices

\(A\) is hermitian congruent to \(P^* A P\)
where \(P\) is invertible.

Complex dot product: \(\bar{x} \cdot \bar{y} = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n\)

\[\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \bar{y} = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{pmatrix}\]

\(\bar{x} \cdot \bar{y} \in \mathbb{R}\) (true for any Hermitian form)
\[ \mathbf{x} \cdot \mathbf{x} \geq 0 \quad \mathbf{x} \cdot \mathbf{x} = 0 \iff \text{only if } \mathbf{x} = \mathbf{0}. \]

positive-definite.

A \underline{unitary matrix} (over \( \mathbb{C} \)) is a complex matrix \( A \) such that \( A^* = A^{-1} \).

For any field \( F \) with involution, any \underline{Hermitean form} \( H \) over \( F \), \( U(H) = \text{group of automorphisms of } H \)

\[ U(\text{dot product on } \mathbb{C}^n) = U_n = \left\{ n \times n \text{ unitary matrices} \right\}. \]
Main axes theorem (matrix form): Let $A$ be a Hermitian matrix. Then

1. all eigenvalues of $A$ are real
2. any two eigenvectors with respect to different eigenvalues are orthogonal with respect to complex dot product.
3. $A$ is diagonalizable.

Corollary: There exists a unitary matrix $P$ such that

$$P^{-1} A P \text{ is diagonal}$$

$$\quad \quad (\overset{=}{p^* A p})$$
Proof of Corollary from Theorem: By the theorem, there exists an orthonormal basis consisting of eigenvectors.

\[ \vec{v}_1, \ldots, \vec{v}_n \]

\[ \vec{v}_i \cdot \vec{v}_j = 1 \quad i = j \]

\[ 0 \quad \text{else} \]

Column vectors in \( \mathbb{C}^n \).

If we put \( \vec{v}_1, \ldots, \vec{v}_n \) together to make a matrix, the matrix will be unitary. Thus, matrix is \( P \). \( \square \)

Note: There is a Sylvester’s law for Hermitean forms, signature = \# positive eigenvalues - \# negative eigenvalues.
Any time a Hamiltonian matrix is Hermitian congruent to a diagonal matrix,

\[ \text{signature} = \# \text{ positive entries} - \# \text{ negative entries}. \]

**Remark about real matrices**

**Corollary**: Let \( A \) be a real symmetric real matrix. Then there exists a real orthogonal matrix \( P \) such that

\[ P^{-1}AP \text{ is diagonal} \]

\[ \begin{pmatrix} \mathbb{1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \mathbb{1} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \]
Proof: Follow the proof in the Hermitian case, because $A$ is real, you can choose the eigenvectors to be real. \(\square\)

Interpretation in terms of forms:

1. Let $H$ be a Hermitian form on $C^n$. Then there exists an isomorphism $\phi: (C^n, H) \rightarrow (C^n, H)$ such that $\phi$ preserves the dot product and $\phi^*H$ has a diagonal matrix.

2. Let $\omega$ be a symmetric bilinear form on $\mathbb{R}^n$. 

Then there exist an isomorphism \( \varphi: (\mathbb{R}^n, \cdot) \rightarrow (\mathbb{R}^n, \cdot) \)
such that \( \varphi \) preserves the dot product and 
\( \varphi^t \) has a diagonal matrix.

Proof of the Theorem: Let \( \mathbf{v} = (v_i) \) be an

eigenvector of a Hermitian matrix \( \mathbf{A} \) with

respect to an eigenvalue \( \lambda \).

\[
\mathbf{v}^* \mathbf{A} \mathbf{v} = \mathbf{v}^* \mathbf{A} \mathbf{v} = \mathbf{v}^* (\mathbf{v}^* \mathbf{v}) \mathbf{v} = \mathbf{v}^* \mathbf{v} \lambda \geq 0
\]

\[
(\mathbf{v}^* \mathbf{A}^t) \mathbf{v} = (\mathbf{A} \mathbf{v})^* \mathbf{v} = (\mathbf{A} \mathbf{v})^* \mathbf{v} = \mathbf{v}^* \mathbf{A} \mathbf{v} = \lambda \mathbf{v}^* \mathbf{v} \geq 0
\]

\[
\therefore \lambda = \lambda^* (\lambda = \bar{\lambda}).
\]
(2) Let \( \vec{v} \) be an eigenvector corresponding to eigenvalue \( \lambda \).

\( \vec{w} \) be an eigenvector corresponding to eigenvalue \( \mu \), \( \lambda \neq \mu \).

\[
\vec{v}^* A \vec{w} = \vec{v}^* \mu \vec{w} = \mu (\vec{w} \cdot \vec{v})
\]

\[
(\vec{v}^* A^*) \vec{w} = (A \vec{v})^* \vec{w} = \sum \vec{v}^* \vec{w} = \lambda (\vec{w} \cdot \vec{v})
\]

\[\lambda \in \mathbb{R}\]

\[
\therefore \, \vec{w} \cdot \vec{v} = 0
\]
3. Proof that $A$ is diagonalizable.

Claim: Let $W$ be the eigenspace of $A$ corresponding to $\lambda \in \mathbb{R}$.

If $\vec{w} \in W$ ( $\vec{w} \cdot \vec{w} = 0$ and $\vec{w} \in W$), then $A \vec{w} \in W$.

Proof: $(A \vec{w}) \cdot \vec{w} = \vec{w}^* A \vec{w} = (\vec{w}^* A^*) \vec{w} =

= (A \vec{w}) \cdot \vec{w} = \lambda \vec{w} \cdot \vec{w} = \lambda \cdot 0 = 0.$

Let $V \subseteq \mathbb{C}^n$ be the subspace spanned by all the eigenvectors. Then by the Claim,
A defines a linear transformation \( \varphi : V^+ \rightarrow V^+ \).

But that has to have an eigenvalue of \( V^+ \neq 0 \):

(choose any basis, \( \varphi \) has matrix \( M \)

eigenvalue of \( M \) is a zero of

\[ \det (xI - M) , \]

exists because \( \mathbb{C} \) is algebraically closed).

\( x^+ \neq 0 \)

so there is an eigenvector \( v \) of \( V^+ \) if \( V^+ \neq 0 \)

contradiction. \( \therefore V^+ = 0. \)
Example: Finding the mean axes of a conic.

\[ 2x^2 + 2xy + 2y^2 = 10 \]

\[
\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 10
\]

Symmetric real matrix

Finding eigenvalues

\[
\det \begin{pmatrix} x - 2 & -1 \\ -1 & x - 2 \end{pmatrix} = x^2 - 4x + 3
\]

\[
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0
\]

\[
x = 1 \quad u = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

\[
x = 3 \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]
The axes are \( (-\frac{1}{2}, \frac{1}{2}) \quad (\frac{1}{2}, \frac{1}{2}) \)

\[
(x - y)^2 + 3(x + y)^2 = 20.
\]

Exam: Monday 1:10 - 3 pm 4088 EH