1. Let

\[ \sigma_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}. \]

(a) Is \( y^n + \sigma_1(x_1, \ldots, x_n)y^{n-1} + \cdots + \sigma_{n-1}(x_1, \ldots, x_n)y + \sigma_n(x_1, \ldots, x_n) \) an irreducible polynomial in \( \mathbb{Z}[x_1, \ldots, x_n, y] \)?

Answer: No, it isn’t. It is equal to

\[ \prod_{i=1}^{n} (y + x_i). \]

(b) Is \( y^n + \sigma_n(x_1, \ldots, x_n)y^{n-1} + \cdots + \sigma_2(x_1, \ldots, x_n)y + \sigma_1(x_1, \ldots, x_n) \) an irreducible polynomial in \( \mathbb{Z}[x_1, \ldots, x_n, y] \)?

Answer: Yes, it is. Apply Eisenstein criterion to the ideal \((x_1, \ldots, x_n)\) in the ring \( \mathbb{Z}[x_1, \ldots, x_n] \). This ideal is prime because the quotient ring is \( \mathbb{Z} \), which is an integral domain.
2. (a) Let $F$ be a field. Describe an algorithm for deciding whether

$$(p_1, \ldots, p_m) \subseteq (q_1, \ldots, q_k)$$

for $p_1, \ldots, p_m, q_1, \ldots, q_k \in F[x_1, \ldots, x_n]$. You may use algorithms covered in class as procedures without describing them.

Answer: Let $g_1, \ldots, g_N$ be the Gröbner basis for $(q_1, \ldots, q_k)$ with respect to some monomial order. The inclusion in question holds if and only if each $p_i$ gives remainder 0 when divided by $(g_1, \ldots, g_N)$ using the general division algorithm.

(b) Is $(x^4 + x^4, x^6 + y^6) \subseteq (x^2 + y^2, xy)$ in $\mathbb{Q}[x, y]$?

Answer: The Gröbner basis of $(x^2 + y^2, xy)$ with respect to the lexicographic order $x > y$ is $x^2 + y^2, xy, y^3$. Applying the algorithm shows that the answer is yes.
3. (a) Describe an algorithm for deciding whether, for elements $b, a_1, \ldots, a_k \in \mathbb{Z}^n$, $b$ is an element of the subgroup generated by $a_1, \ldots, a_k$. You may use algorithms covered in class as procedures without describing them.
Answer: Suppose we write $a_1, \ldots, a_k, b$ as columns. Then they form an $n \times (k + 1)$-matrix. Apply equivalent row and column operations on the matrix $a_1, \ldots, a_k$ to reduce it to Smith canonical form, while also applying all the elementary row operations to $b$ simultaneously (no column operations on $b$ are allowed). In the resulting matrix, the answer to our question is positive if and only if the number in the last column in each row is divisible by the diagonal term in the same row. (To justify this, in this notation, column operations change the generators of $\langle a_1, \ldots, a_k \rangle$, while row operations change the free generators of $\mathbb{Z}^n$.)

(b) Does the system of equations
\[
2x + 8y + 4z = 2, \\
16x + 8y + z = 5
\]
have a solution with $x, y, z \in \mathbb{Z}$?
Answer: Applying the algorithm, one finds the answer is yes.
4. Prove that if $A, B$ are square matrices over a field $F$ and $F$ is a subfield of another field $K$, then $A, B$ are similar over $F$ if and only if they are similar over $K$. [You may use theorems covered in class only without proof.]

Answer: The rational canonical (or Smith canonical) form of $A$ and over $F$ is also a rational canonical (or Smith canonical) form over $K$, so by uniqueness, the rational canonical (or Smith canonical) forms of $A$ over $F$ and $K$ coincide. Similarly for $B$. Thus, if the rational canonical (or Smith canonical) forms of $A, B$ over $K$ coincide, the (identical) rational (or Smith) canonical forms over $F$ also coincide.
5. Find a matrix in generalized Jordan canonical form over \( \mathbb{R} \) similar to the matrix

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

Answer: The matrix is associated to the polynomial \((x^3 + 1)^2\). Factoring the polynomial into irreducible factors over \( \mathbb{R} \), one finds that the generalized Jordan form over \( \mathbb{R} \) is

\[
\begin{pmatrix}
-1 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}.
\]