A is an abelian group.

Most of the time, we talk about cochain complexes

\[ \cdots \leftarrow C \leftarrow C \leftarrow C \leftarrow C \leftarrow \cdots \]
\[ C^m(X;A) := \text{Hom}(C_n(X), A) \]

then form a co-chain complex \[ C^*(X;A) \]

\[ (C(X;A) = C_*(X;A)) \]

Comment 1: \[ \text{Hom}(\bigoplus S, A) = \prod_{S} A \]

Now generally,

\[ \text{Hom}(\bigoplus B_i, A) = \prod_{\mu \in I} \text{Hom}(B_{\mu}, A) \]

Therefore, \[ C^m(X;A) = \left\{ \sum_{\sigma : \Delta^m \rightarrow X} a_{\sigma} \cdot \sigma \mid a_{\sigma} \in A \right\} \]
Write an explicit formula for
\[ d^n : C^n(X; A) \to C^{n+1}(X; A) \]
\[ d^n \left( \sum_{\sigma : \Delta^n \to X} a_{\sigma} \cdot \sigma \right) = \ldots \]

This is a typical pattern: finite sums are covariant, infinite sums are contravariant (= uncostructured). (also occurs in functional analysis)

"middle ground: Hilbert spaces"
Why cohomology? One reason is that cohomology was known first. (Other methods for defining (co)homology are more geometric.) Techniques of defining (co)homology are techniques of defining integration.

In fact, (co)homology is the "qualitative aspect of integration." Smooth k-dimensional integration in n-space (n-manifolds) of differential forms: oriented compact k-submanifold $K$ of an $n$-manifold $M$. What can we integrate...
over $K$ without any further data (such as a measure?) \[ \int \omega \uparrow \]

A $k$-form on $M$: look at the tangent space at $x \in M$: $TM_x$. Take the dual: $TM_x^* \cong \text{Hom}(TM_x, \mathbb{R})$.

$\omega: x \mapsto \omega(x) \in \wedge^k TM_x^*$

\[ \uparrow \]

product of $k \otimes TM_x^+$

... \otimes \sigma \otimes ... \otimes \sigma \otimes ...
\[
\omega \in \operatorname{smooth} (\text{locally, everything ultimately happens in } \mathbb{R}^n).
\]

Why exterior power? $k$-volume of a $k$-parallelpiped determined by \( v_1, \ldots, v_k \in \mathbb{R}^n \):

\[
\| v_1 \wedge \cdots \wedge v_k \| \leq \wedge^k \mathbb{R}^n
\]

\( e_1, \ldots, e_n \)

\( e_1 < \cdots < e_n \)

As an orthonormal basis.

Apply chain rule to integrate.
Differential forms are fully functorially contravariant w.r.t. smooth maps:

\[ f: M \to N \text{ smooth} \]

\[ f^*: \Omega^k N \to \Omega^k M \]

The base point \( \ast \) for 1-forms is for smooth sections of \( T\ast N \).

\[ f^*: T\ast N \to T\ast f(M) \]

\[ \text{Note: vector fields are not functorial in either direction.} \]

\[ \text{Smooth functions are 0-forms. They are contravariant.} \]
\[ M \xrightarrow{f} N \xrightarrow{h} \mathbb{R} \]

\[ f^* : C^\infty(N) \rightarrow C^\infty(M) \]

\[ Df_x : T M_x \rightarrow TN_{f(x)} \]

\[ (Df_x)^* : TN_{f(x)} \rightarrow TM_x \]

\[ \mathbb{C}^k \xrightarrow{z} \text{Hom}(\mathbb{C}^k, \mathbb{R}) \]

\[ \omega \in \Omega^k M \]

\[ \mathbb{R} \xrightarrow{u} M \]

\[ U \]

The total differential (covariant) is both controversial and controversial now.
\( f^*(\omega) \in \mathcal{C}^k(U) \quad f \text{ smooth} \)

\[
\begin{cases}
\uparrow \quad \text{open} \\
\text{integral only depends on the image and orientation.}
\end{cases}
\]

\[ k \mapsto \Lambda^k (T^* \mathbb{R}^n) \bigg|_x = \Lambda^k \mathbb{R}^k = \mathbb{R} \]

A function, we can integrate with respect to the Lebesgue measure.

\[ \mathbb{R}^k \rightarrow \mathbb{R}^k \]

\[ U \quad \text{this integral is invariant under pullback } \mathbb{R}^k \rightarrow \mathbb{R}^k \]
by the substitution theorem (the Jacobian is the determinant of $Df$ is the top exterior product.

(Ke, Pulver: Introduction to Nonlinear Analysis — Electronic access)

There is a differential on differential forms:

$$d = d^k : \bigwedge^k (\mathcal{M}) \to \bigwedge^{k+1} (\mathcal{M})$$

(a generalization of $d^\star$, grad, curl)
\[ \dim \wedge^n \mathbb{R}^n = \binom{n}{k} \quad \text{for} \quad 3 \leq \dim \leq 1 \]

\[ \mathbb{R}^3 \quad \mathbb{R}^3 \]

In local coordinates, \( x_1, \ldots, x_m : U \rightarrow V \subseteq \mathbb{R}^n \)

\[ d \left( h \, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right) = \]

\[ = \sum_{i=1}^{m} \left( \frac{\partial h}{\partial x_{i_j}} dx_{i_j} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right) \]

\[ \text{df} : \mathbb{R}^m \rightarrow \mathbb{R}^n \]

\[ df_x = Df \]

includes Hodge *- operator - duality
\[ \mathbb{R}^n \cong (T R^*_x) \]

\[ \text{permutations handle signs!} \]

\[ \begin{align*}
\text{div} & = 0 \\
\text{curl grad} & = 0
\end{align*} \]

\[ H^k \left( \Omega^0 (\mathbb{R}) \right) \xrightarrow{d} \Omega^1 (\mathbb{R}) \rightarrow \cdots \rightarrow \Omega^n (\mathbb{R}) = \]

\[ =: H^k_{\text{DR}} (\mathbb{R}) \]

\[ \Rightarrow \text{De Rham cohomology} \]
Theorem: (The topological de Rham theorem): There is a canonical isomorphism

\[ H^k_{dR}(M) \cong H^k(M; \mathbb{R}) \]

singularity cohomology

(Caution: In algebraic geometry, these symbols may mean something else, you might need \( C \))

p-adically, this is deep \( B_{dR} \)

Sketch proof of topological de Rham theorem:

We connect smooth, singular complexes smoothly
\[\delta : \Delta_k \rightarrow M\]

\[C^\text{smooth}_k (M; \mathbb{R}) \rightarrow H^k_{\text{smooth}} (M; \mathbb{R})\]

\[H_k (M; \mathbb{R}) \leq \text{cyclicly shown}\]

\[H^k (\bullet; \mathbb{R}) \cong \text{Hom} (H_k (\bullet; \mathbb{R}), \mathbb{R})\]

\[\int : \bigotimes_{\mathbb{R}} C^\text{smooth}_k (M; \mathbb{R}) \rightarrow \mathbb{R}\]

\[\int_{M} \mid \llcorner\]

\[\mathcal{L}^k (M) \rightarrow C^\text{smooth}_k (M; \mathbb{R})\]
Next, this proves differential
\[ \nabla \cdot dw = d(\omega) \]
\[ \int_\gamma (\omega) = \int_0^1 dw \leq \text{Stokes theorem.} \]
(Cauchy's Principle Lemma \quad H^1(\mathbb{R}^n) = 0 \quad \forall \neq 0, \]
Mayer-Vietoris)