The homology (with coefficients) and cohomology of a $CW$-complex

Supplement homology coboundary axiom with a coproduct axiom

$$\bigoplus_{i \in I} H_n(X_i; A) \to H_n\left( \coprod_{i \in I} X_i; A \right).$$

(See 5.9.2 notes – contents of each class now listed! on my web page.)
let \( \emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \)

be a CW complex. First study

\[ H_k(X_m, X_{m-1}; A) \quad \text{and} \quad H^k(X_m, X_{m-1}; A). \]

(A technical comment: let \((X, \mathcal{I})\) be a pair.

Then \( C_k(X, \mathcal{I}) \) is a free abelian group.

\[ \text{the free abelian group on } \{ \sigma : \Delta^n \rightarrow X \mid \sigma(\Delta^n) \notin \mathcal{I} \} \]

HW: Prove that if \( S \subseteq T \) then \( \mathbb{Z}S \subseteq \mathbb{Z}T \)

and \( \mathbb{Z}T / \mathbb{Z}S \cong \mathbb{Z}(T \setminus S) \).
This is important for the exactness axiom in homology with coefficients and cohomology.

\[ 0 \to C(Y) \to C(X) \to C(X,Y) \to 0 \]

\[ \otimes A : 0 \to C(Y; A) \to C(X; A) \to C(X,Y; A) \to 0 \]

In general, a short exact sequence \( \otimes A \)

\[ \cdots \to A \to 0 \]

(right exactness). Example:

\[ 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0 \]

\[ \otimes \mathbb{Z}/2 \]

\[ \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{\xi} \mathbb{Z}/2 \to 0 \]
In our case, $\Theta$, the difficulty will not occur

- exactness is measured on each $C_k$ separately

(terminology: level, dimension, degree)

But on each $C_k$, we have a short exact sequence of free abelian groups, which must split:

\[ 0 \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot \rightarrow 0 \]

\[ \text{lift on generators, by freeness, extends} \]

A preserves coproducts, hence split.
Short exact sequences.

With \( \text{Hom} (\mathbb{Z}, A) \), the story is the same.

\[
0 \to \mathbb{Z}_1 \to \mathbb{Z}_2 \to \mathbb{Z}_3 \to 0
\]

is short exact.

\[
\text{Hom}(\mathbb{Z}_1, A) \leq \text{Hom}(\mathbb{Z}_2, A) \leq \text{Hom}(\mathbb{Z}_3, A) \leq 0
\]

\[
0 \to \mathbb{Z} \overset{2}{\to} \mathbb{Z} \to \mathbb{Z}/2 \to 0
\]

\[
\text{Hom}(\mathbb{Z}, \mathbb{Z}/2) : \quad \mathbb{Z}/2 \cong \mathbb{Z}/2 \cong \mathbb{Z}/2 \cong 0.
\]

Back to our CW complex \( X \):

\[
H_n (X_n, X_{n-1}; A) = \bigoplus_{i \in \text{Im}} A = \mathbb{Z} \left[ \text{Im} \right] \otimes A
\]
\[ H^k(X_n, X_{n-1}; A) = \bigoplus_{\sigma \in \text{In}_n} A = \text{Hom}(\mathbb{Z}[\text{In}_n], A) \]

This is just excision: "pinpoint the n-cells of \( X_n \) (remove a point from the interior of each) to obtain \( X_n \approx X_{n-1} \)."

Use deformation retract.

Hence:
\[ H^k(X_n, X_{n-1}; A) = H^e_0(X_n, X_{n-1}; A) \]

\[ \bigoplus_{\sigma \in \text{In}_n} X_n \]

Excision:
\[ H^e_n(X_n \setminus X_{n-1}, X_n \setminus X_{n-1}; A) \]
\[ H_k(D^n, S^{n-1}; \text{another point}; A) \]

\[ \bigoplus_{i \in I_n} H_k(D^n, D^n - i; \text{interior point}; A) \]

\[ I_n = \text{the set of n-cells} \]

\[ H_k(C(D^n, S^{n-1}; A) = A \quad k = n \]

\[ 0 \quad \text{else} \]

The case of cohomology is similar.

This computation suggests making a chain complex

\[ \text{Cell}(X) = \text{coeffs. are in } \mathbb{Z} \]
\( C_{\text{cell}} (X ; A) \) := \( C_{\text{cell}} (X) \otimes A \)

\( C^*_{\text{cell}} (X ; A) \) := \( \text{Hom} (C_{\text{cell}} (X), A) \)

\( C^m_{\text{cell}} (X) \) := \( H_m (X_n, X_{n-1} ; Z) = Z [I_m] \)

\( C^m_{\text{cell}} (X ; A) \) := \( H_m (X_n, X_{n-1} ; A) = Z [I_m] \otimes A \)

\( C^m_{\text{cell}} (X ; A) \) := \( H_m (X_n, X_{n-1} ; A) = \text{Hom} (Z [I_m], A) \)

We need to define the differential. The connecting map of the long exact sequence of \( (C(X_n, X_{n-1}, X_{n-2}) \) is:

\( d_m : H_m (X_n, X_{n-1} ; Z) \rightarrow H_{m-1} (X_{n-1}, X_{n-2} ; Z) \).
Alternately,

\[ H_n(X_m, X_{m-1}; \mathbb{Z}) \longrightarrow \cdots \longrightarrow H_n(X_{m-1}, X_{m-2}; \mathbb{Z}) \]

To prove that \( d_{n-1} \circ d_n \) is zero:

\[ H_n(X_m, X_{m-1}; \mathbb{Z}) \xrightarrow{d_n} H_{n-1}(X_{m-1}, X_{m-2}; \mathbb{Z}) \xrightarrow{d_{n-1}} H_{n-2}(X_{m-2}, X_{m-3}; \mathbb{Z}) \]
The composition is of these are consecutive maps in the LES in homology for 

\[(X_{n+1}, X_{n-1}).\]

This way, we defined \(C^+_{\text{cell}}(X; A)\), \(C^\ast_{\text{cell}}(X; A)\).

(Example: A simplicial complex is a CW-complex.
In the case of \(X\) a simplicial complex, this is simplicial homology and cohomology).

Theorem: \(H_n\left( C^\ast_{\text{cell}}(X; A) \right) \cong H_n(X; A) \)

\(H^n\left( C^\ast_{\text{cell}}(X; A) \right) \cong H^n(X; A)\). \(\square\)
Comment: This also works for pairs. What is a CW pair? (More general than a CW complex.)

A CW pair is defined the same as a $(X, Y)$.

$\ast$ CW-complex $X$ except $: X_{-1} = Y$.

The differential in CW-$(\infty)$ homology:

It suffices to do the case of homology with coefficients in $\mathbb{Z}$ (because the other cases are $\mathbb{Z} \otimes A, \text{Hom}(\mathbb{Z}, A)$).
This is called the degree of a map.

How do we compute the effect of $f_{n-1}$ at a point $x$?

$$f_{n-1}(x) = f_{n-1}(y_{n-1}) + \epsilon_{n-1}x$$

$$S_{n-1} = \sum_{i=1}^{n-1} X_{n-1}/X_{n-2} = V_{n-1}$$
See 592 notes.