The differential in $\text{CW}$-homology is more precisely on $\text{Cell}(X)$ with coefficients in $\mathbb{Z}$

$(\partial A) \in \text{Hom}(\mathbb{Z}, A)$ - all the other cases

(see Section 2 notes)

$X$ $\text{CW}$-complex

$H_n(X_{m+1}, X_{m-1}) \rightarrow H_{n-1}(X_{m-1}, X_{m-2})$

$\partial \rightarrow H_{n-1}(X_{m-1}) \rightarrow H_{n-1}(\leq)$
work on one cell in \( X^n \)

\[(0^n, s^{n-1}) \to (X_m, X_{m-1})\]

\[H_m(0^n, s^{n-1}) \to H_m(X_m, X_{m-1})\]

\[\partial \mid \quad \partial\]

\[H_{m-1}(s^{n-1}) \to H_{m-1}(X_{m-1})\]

\[\text{dim.} \quad m \geq 1\]

(0 requires extra care, e.g., \( \widetilde{H}_1 \))
$S^{n-1} \to X_{n-1} \to X_{n-1}/X_{n-2} = \bigvee S^{n-1} \to S^{n-1}$

attachment
map of $i$-cell

map of $j$-cell

Thing of the differential as a $I_{m-1} \times I_m$-matrix

$(V_j \subset I_m \ni f_j \subset I_{m-1})$ finite

$a_{ij} = 0$ unless $i \in \{j, \bar{j}\}$

$a_{ij}$ is the effect of $\delta_{ij}$ in homology

$H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$

$\mathbb{Z}$

$\mathbb{Z}$
multiplication by the number \( a_{ij} \in \mathbb{Z} \)

The number is called the \textit{degree} \( \deg f_{ij} \).

It can be computed geometrically. (Caution: the isomorphisms depend on orientation of \( S^{n-1} \). The orientation is arbitrary, except keep it fixed in each cell.)

Orientation of a smooth manifold \( M \) is the orientation of \( T_{x} M \) at every \( x \in M \), to be called \textit{canonical}.

orientation of an \( n \)-vector space \( V \), \( \text{dim } V = n \)
is picking a connected component
of \( \text{Iso}(V, \mathbb{R}^n) \),

\[ \cong \text{Gl}_n(\mathbb{R}) \] (has 2 connected components given by the

sign of the determinant)

If \( f : S^n \to S^n \), \( f \sim g \), \( y \in S^n \) with an open

neighborhood \( U \) such that \( f|_{g^{-1}(U)} \) is smooth,

for every \( x \in g^{-1}(y) \), \( Dg|_x : TS^n_x \to TS^n_y \) is an \( \equiv \).

Then define:
Theorem: \( \text{deg } f = \sum \text{sgn } Dg_x \)

\[ x \in g^{-1}(y) \]

+1 if \( Dg_x \) preserves orientation

-1 if \( Dg_x \) reverses orientation. \( \square \)

(See 592)

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Example: \( \mathbb{R}P^m \), \( \mathbb{R}P^\infty \).

Representing \( \mathbb{R}P^n \) as a CW-complex.

\[ \mathbb{R}P^n = S^n / \sim -x \sim x \subset \text{quotient topology} \]
$S^n = \{(x_0, \ldots, x_n) \mid \sum x_k^2 = 1\}$

Unit sphere in $\mathbb{R}^{n+1}$.

$S^{n+1} = \{(x_0, \ldots, x_n) \in S^n \mid x_n \geq 0\}$

After identification, $S^n$ becomes $\mathbb{C}P^n$ cell.

$S^{n-1} \subset S^n$

$S^{n-1} = \{(x_0, \ldots, x_{n-1}, 0) \in S^n\}$

This becomes $\mathbb{R}P^{n-1}$
The attaching map is the covering map
\[ s^{m-1} \to \mathbb{R}P^{m-1} = \mathbb{S}^{m-1} / \sim \]
\[ \mathbb{R}P^m = \bigcup \left( \cdots \subset \mathbb{R}P^{m-1} \subset \mathbb{R}P^m \subset \cdots \right) \]

is also a CW complex, with 1 cell in every dimension \( \geq 0 \).

Comment: Proving rigorously that the CW complex \( X_m \) constructed really is homeomorphic to \( \mathbb{R}P^m \):

**Step 1:** \( X_m \to \mathbb{R}P^m \) by a bijective continuous map.

because \( X_m \) is successively
constructed from $X_{n-1}$ by a
functor

\[ S^{n-1} \rightarrow X_{n-1} \rightarrow \mathbb{R}P^{n-1} \]

which has a universal
property

Step 2: If $f: X \rightarrow Y$ is a bijective
continuous map where $X$ is compact and
$Y$ is Hausdorff, then $f$ is a homeomorphism.
Computing $H_k(\mathbb{R}P^n; A)$, $H^k(\mathbb{R}P^n; A)$

HW: Compute

$H_k(\mathbb{R}P^\infty; A)$

$H^k(\mathbb{R}P^\infty; A)$

$\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z} \to \mathbb{Z}$

$d^\text{cell} = 2 \quad n \text{ even}$

$0 \quad n \text{ odd}$

degree of $f$

$\mathbb{R}P^{n-1} \to \mathbb{R}P^n$
\[ H^k(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2 & k \text{ odd } 0 < k < n \\ 0 & \text{otherwise} \end{cases} \]

\[ H^k(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0 \text{ or } k = n \\ \mathbb{Z}/2 & k \text{ odd } 0 < k < n \\ 0 & \text{otherwise} \end{cases} \]

Now look at \[ H^k(\mathbb{RP}^n; \mathbb{Z}) : \]
\[ \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \]
\[ \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \]
\[ \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \cdots \]
\[ \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \cdots \]
\[ \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \]
\[ \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \]
\[ H^k(\mathbb{R}P^\infty; \mathbb{Z}) = \mathbb{Z} \quad k = 0 \quad \text{or} \quad (k = n \text{ and } n \text{ odd}) \]

\[ \forall k \quad k \text{ even} \quad 0 < k < n \]

\[ 0 \quad \text{otherwise} \]

\[ H_k(\mathbb{R}P^n; \mathbb{Z}/2) \quad \mathbb{Z}/2 \overset{0}{\rightarrow} \mathbb{Z}/2 \cdots \overset{0}{\rightarrow} \mathbb{Z}/2 \]

\[ = H^k(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2 \quad k \quad 0 \leq k \leq n \]

\[ = 0 \quad \text{else} \]

\[ H_k(\mathbb{R}P^n; \mathbb{Q}) \quad \mathbb{Q} \overset{2}{\rightarrow} \mathbb{Q} \cdots \overset{2}{\rightarrow} \mathbb{Q} \overset{0}{\rightarrow} \mathbb{Q} \quad n \text{ even} \]

\[ = \mathbb{Q} \quad k = 0 \quad n \quad (k = n \text{ odd}) \]

\[ \mathbb{Q} \overset{0}{\rightarrow} \mathbb{Q} \overset{2}{\rightarrow} \cdots \overset{2}{\rightarrow} \mathbb{Q} \overset{0}{\rightarrow} \mathbb{Q} \quad n \text{ odd} \]
$0 \in \mathcal{H}_n(\mathbb{R}^n; \mathbb{Q})$ for $n \in \mathbb{Q}$.

Something is happening when I am changing coefficients, and/or taking cohomology.

Next: The universal coefficient theorem.

* HW: The lens space for $\mathbb{C}/\mathbb{R}$:

$$S^{2n-1} = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum |z_k|^2 = 1 \}$$
Now \( S' = \{ \lambda \in \mathbb{C} \mid \lambda\tau = 1 \} \), a group, acts on \( S^{2n-1} \).

\[ d \cdot (z_1, \ldots, z_n) = (\lambda^2 z_1, \ldots, \lambda^2 z_n). \]

Consider \( \mathbb{Z}/k \subset S' \) (a subgroup) and put

\[ L^k : = S^{2n-1} \bigg| _{z \sim \lambda \in \mathbb{Z}/k} \quad \lambda \in \mathbb{Z}/k, \quad z \in S^{2n-1} \]

Put a quotient topology.

(2) Find a CW-structure on \( L^k \).

(Hint: even and odd - dimensional)
Cells are different.) Consider dim. 1, 2.

(b) Compute $H_i(C^n; \mathbb{Z})$, $H_i(C^*; \mathbb{Z})$. 