I have been called to jury duty! on Monday Sep 29.

If I have to appear, I will send an email to the class, in that case, there is no class on Monday. If I do not have to appear, I will also send an email to the whole class, and in that case, there will be a class on Monday.

Category: Chain complexes of $R$-modules, $R$-Chain chain maps of $R$-modules (commutative case)
A free (projective) resolution of an \( R \)-module \( M \) is a chain complex of free \( R \)-modules \( C \) such that:

1. \( C_m = 0 \) for \( m < 0 \)
2. \( H_k C = 0 \) for \( k \neq 0 \)

where \( H_0 C \cong M \). This is a given homomorphism.

**Lemma:** A free resolution of an \( R \)-module \( M \) always exists.
Proof: 

The kernel is 

\[ 0 \rightarrow M_{n+1} \rightarrow R(CM_n) \rightarrow M_n \rightarrow 0 \]

\[ \varphi \quad (m) \mapsto m \in M \]

\[ M_{n+1} \cong M \]
A long exact sequence:

\[ \cdots \rightarrow R[N_i] \rightarrow R[N_{i+1}] \rightarrow R[N_0] \rightarrow M \rightarrow 0 \]

An augmented resolution of \( M \).

(Not a resolution)

Free resolution after getting rid of \( M \). \[ \square \]
Remark: The free resolution is a functor

\[ \text{R-Mod} \rightarrow \text{R-Chain} \]

(Therefore, from now on, \( \text{Tor}_n(R, M) \), \( \text{Ext}_n^R(M, N) \) are functors!)

We say \( R \) has \text{projective dimension} \( m \)
if there is always an \( m \)-term projective resolution of an \( R \)-module.

It coincides with \text{Krull dimension}
for regular rings: maximal length
of a chain of prime ideals under
inclusion +1. \((\dim(fv(c)) = 0)\).

**Chain homotopy**: look at 592 notes on the homotopy axiom. If \(f, g: C \to D\) are chain maps, a chain homotopy

\[ h: f \approx g \]

is a sequence of homomorphisms

\[ h_n: C_n \to D_{n+1} \]

such that

\[ h_n \cdot d + d h_n = f_n - g_n \]

\[ h: \text{the hom} \]
can define a chain homotopy in $R$-modules.

Chains-homotopic maps of $R$-modules induce the same thing in homology.

Also, note that $\otimes_R N$, $\text{Hom}_R(?, N)$ preserve chain homotopy.

\[ H/W : \text{Prove the hom - case} \]

\[ f = g \implies Hom_R(f, N) = Hom_R(g, N) \]

in $R$-chain
Proposition: Let \( f : M_1 \to M_2 \) be a homomorphism of \( R \)-modules and let \( C_1 \) be projective resolutions of \( M_1 \). Then there exist a morphism in \( R \)-chains \( F : C_1 \to C_2 \) which induces \( f \) on homology. Moreover, \( F \) is unique up to chain homotopy.

Remark: (\( C_2 \) does not have to be projective).

Other models: "it is easy to map from something free to something acyclic - 592"
The relevance of the Proposition: The Proposition fails to test the correctness of the definition of $\text{Tor}_R^n$ and $\text{Ext}_R^n$.

Let $ar{C}$ be the functorial free resolution of $M$ constructed above. (We defined $\bar{C} \text{Tor}_R^n (M, N) = \text{Tor}_R^n (M, N)$, where

$$\text{Tor}_R^n (M, N) = \text{Tor}_R^n (C \otimes_R N),$$

and denote by $\text{Ext}_R^n (M, N) = H^n (\text{Hom}_R (C, N))$.}
Any two choices of $\psi$ are homotopic.
hence the corresponding choices of $\phi_{\mathbb{P}^1}$ are homotopic. \[ \therefore \phi \] (which comes from $\phi$) is always the same.

\[ \text{(HW)}: \Lambda_{\mathbb{Q}}[x] := \mathbb{Q}[x]/(x^2) \]

Find a free $\Lambda_{\mathbb{Q}}[x]$-resolution of $\mathcal{O}_x$, $x$ act by 0.

Compute $\text{Tor}_n^{\Lambda_{\mathbb{Q}}[x]}(\mathcal{O}_x, \mathcal{O}_x)$, $\text{Ext}^{\infty}_{\Lambda_{\mathbb{Q}}[x]}(\mathcal{O}_x, \mathcal{O}_x)$. 