School: \text{Hom} \ (C, D)

\text{Rounded below}

\text{Rounded above}

\text{Describerum: Identify chain maps} \ C \to D.

\text{from chain to structure vs as cycles.}

If \ f: C_m \to D_n \text{ you let}

3f(\mathbf{x}) = f(d\mathbf{x})

\overline{\delta}f(\mathbf{x}) = d(f(\mathbf{x})).
Total differential
bidegree: \((-m, m)\)

\[ f(dx) + (-1)^m df(x) = 0 \]

is not the correct equation of a chain map.

The fix:

\[ \partial f(x) = \int (dx) (-1)^{m+1} \]

\[ df(x) = df(f(x)) \]

Now the equation becomes

\[ (-1)^{m+1} \int (dx) + (-1)^m df(x) = 0 \]

\(\nabla\) This is the equation of a chain map.
Theorem: With the definition \( \mathcal{D} \), we still have a (canonical) natural iso

\[
\text{Hom}(C \otimes D, E) \xrightarrow{\sim} \text{Hom}(C, \text{Hom}(D, E))
\]

for chain complexes \( C, D, E \), \( C, D \) bounded below, \( E \) bounded above.

Proof (reconciliation of signs):

Canonical map

\[
\bigotimes \quad \text{Hom}(C_m \otimes D_n, E_p) \xrightarrow{\sim} \text{Hom}(C_m, \text{Hom}(D_n, E_p))
\]
of differentials: \((-1)^{m+m+1}\) \(\frac{m+1}{(-1)}\) 
\((m+m+1): m\) 
\((-1)^m\) 
\((-1)^m\) 
\((-1)^{m+1} + m\) 
Hom double chain 
\((-1)^{m+m+1}\) 
\((-1)^m\) 
multiply by \((-1)^{m+m}\) 
\((-1)^m\) 
\((-1)^m\) 

(Heuristics: \(C^* = \text{Hom}(C, \mathbb{Z})\)) 
\((C \otimes D)^* = D^* \otimes C^*\).
The cup product:

To do homology theory, work in chain complexes up to chain homotopy, take homology only in the end.

\[ C^*(X) \otimes C^*(X) \rightarrow C^*(X) \]

By above theorem, natural, canonical associative, commutative

\[ T: A \otimes B \rightarrow D \otimes A \]
\[ \mathbb{C}(X) \otimes \mathbb{C}(X) \otimes \mathbb{C}(X) \rightarrow \mathbb{Z} \]

I have:

\[ \rho: \mathbb{C}^\ast(X) \otimes \mathbb{C}^\ast(X) \rightarrow \mathbb{Z} \]

by Theorem, \( \hom(\mathbb{C}^\ast(X), \mathbb{Z}) \)

\[ \mathbb{C}^\ast(X) \otimes \mathbb{C}^\ast(X) \otimes \mathbb{C}^\ast(X) \]

\[ \downarrow \text{Id} \otimes \text{Id} \otimes \mathbb{C}^\ast(\emptyset) \]

\[ \mathbb{C}^\ast(X) \otimes \mathbb{C}^\ast(X) \otimes \mathbb{C}^\ast(X \times X) \]
\[
\downarrow \text{Id} \circ \text{Id} \circ \eta
\]
\[
C^*(X) \otimes C^*(X) \otimes C_*(X) \otimes C_*(X)
\]
\[
\downarrow \otimes T \otimes \cdot \downarrow
\]
\[
C^*(X) \otimes C_*(X) \otimes C^*(X) \otimes C_*(X)
\]
\[
\downarrow \text{ev} \otimes \text{ev}
\]
\[
\mathbb{Z}
\]
$u : C^*(X) \otimes C^*(X) \rightarrow C^*(X)$

$h + l = m$

$H^m (C^*(X) \otimes C^*(X)) \rightarrow H^m C^*(X)$

$H^k (C^*(X)) \otimes H^l (C^*(X))$ \quad \Rightarrow \quad H^m X$

$H_k (X) \otimes H_l (X)$

for elements $c, d$

$H_{k+c} (C \otimes C) \rightarrow H_{k+l+c} (C \otimes C)$

$[c] \otimes [d] \mapsto [c \circ d']$
\[ \begin{align*}
\\text{c, c'} \text{ cycles} \\
\text{If, say, c = du} & \Rightarrow \text{c} \otimes \text{c'} = d(u \otimes c') \\
\text{c' = dv} & \Rightarrow \text{c} \otimes \text{c'} = (-1)^{\deg c} d(c \otimes v).
\end{align*} \]

**HW:** Prove that the following is the limit of the cup product:

\[ C^0(X) \xrightarrow{\delta} \mathbb{Z} \]

\[ \{ \sum k_n x_n \mapsto \sum k_n \} \in C^0(X) \]

\[ \sum_{x_n \in X} \]
Commuteativity and associativity

\[(\Delta, \Delta) : X \to X \times X \]

\[\Delta : X \to X \times X\]

\[\Delta \times \Delta : X \times X \to X \times (X \times X)\]

\[(X \times X) \times (X \times X) \to X \times X \times X \times X\]

\[\text{co-unit} : X \to X, \quad \text{co-associativity}\]

\[\text{HW}: \text{Write down the diagram of co-associativity.}\]
I would like to prove that $H^*(\mathbb{R}P^\infty, \mathbb{F}_2) = \mathbb{F}_2[x]$, where $x = 1$.

The Thom isomorphism (Spanier-Whitehead duality $\Rightarrow$ Poincaré duality)

Vector bundles (Example: The tangent bundle of a manifold)

Loosely speaking, locally trivial families of vector spaces indexed by a topological space $X$. 
Definition of $\pi$-bundle: We have a total space $E$, base projection $\rho : E \rightarrow X$. For any $x \in X$, we are given an open neighborhood $U_x$ of $x$ in $X$ and a homeomorphism

$$\rho^{-1}(U_x) \overset{\cong}{\longrightarrow} U_x \times \mathbb{R}^m$$

On $U_x \cap U_y$, the following is required:
\[(U_x \cap U_y) \times \mathbb{R}^n \cong p^{-1}(U_x \cap U_y) \cong (U_x \cap U_y) \times \mathbb{R}^n\]

The map \(h_y \circ h_x^{-1} : (U_x \cap U_y) \times \mathbb{R}^n \to (U_x \cap U_y) \times \mathbb{R}^n\) is a "linear isomorphism on fibers". There exists a continuous map

\[f : U_x \cap U_y \to \text{GL}_n(\mathbb{R})\]

such that

\[h_y \circ h_x^{-1}(z, u) = (z, (f(z))(u))\]

\[z \in U_x \cap U_y, \quad u \in \mathbb{R}^n \quad \text{GL}_n(\mathbb{R}).\]
If I replace \( U_x \) by a smaller open neighborhood in each \( x \) and the data is the same, I consider it the same bundle.