Approximation theorems for spaces $n \in \{0, 1, \ldots \}$

1. For every space $X$, there exists an $\leq n$-dimensional CW complex $X'_n$ and an $n$-equivalence $f_n : X'_n \to X$.

Definition: Let $X, Y$ be CW complexes. A CW-map $f : X \to Y$ is a (continuous) map
such that \( f(X_m) = Y_m \). Then

(Note: For example, the cell \( (2) \) is functorial in \( CW \)-maps).

(2) Every continuous map of \( CW \)-complexes (or pairs) is homotopic to a \( CW \)-map.

Sketch proof of (2): Let

\[ f : X \to Y \]

be a continuous map, \( f|X_{m-1} \) be \( CW \).

The strategy is to homotopy \( f|_1 \), rel \( X_{m-1} \), to a
may induce $f'$ which is CW when restricted to $X_n$.

Because inclusion of a skeleton is a cofibration, it suffices to construct $f'$ on $X_n$. (After that, just (and the homotopy)

extend the homotopy.)

$X_n$ is obtained from $X_{n-1}$ by attaching $n$-cells.

WHO $G$, there is only one $n$-cell (we can treat every $n$-cell separately by the colimit properties).

WHO $G$, $X_{n-1} = S^{n-1}$

$X_n = D^n.$

\[ S^{n-1} \rightarrow V_{n-1} \text{ CW complex} \]
Let $S^{m-1}$

$S^{m-1} \rightarrow \gamma_{m-1} \subseteq \Delta^{m-1}$

Now that $f \preceq f'$ rel $S^{m-1}$

where $f'(0^m) \subseteq \gamma_m$.

\[ \Box \]

Proof:

\[ \text{Assume this is an } (m-1) \text{-equivalence} \]

\[ X_{m-1} \rightarrow X \]

\[ ? \quad \Delta^{m-1} \quad ? \]

\[ X_m \quad ? \quad m \text{-equivalence} \]

(at least one)
What if $n = 0$? Choose a point in each path component of $X$.

What if $n = 1$? If you chose more than one point in each path component, attach a 1-cell to connect them, (i.e. $\Xi$ on $\pi_0$). To be onto on $\pi_1$, attach a bouquet of ($S^1$)'s generating on each path component $\pi_1$ of each path component of $X$.

What if $n \geq 2$? We may assume $X$ is path-connected.
Attach \( n \)-cells to kill any relations in \( \pi_{m-1} X \).

(more precisely speaking, \( \ker (\pi_m X \to \pi_{m-1} X) \)).

and a bouquet of \( n \)-cycles to be onto on \( \pi_m \).

does this really make \( \prod_k X'_m \to \prod_k X \)

are on \( k \leq m-1 \)?

But for \( k < m-1 \), \( \prod_k X'_m \to \prod_k X_m \)

is \( \cong \) by (2).

Similarly for \( k = m-1 \) to prove \( \prod_k X'_m \to \prod_k X'_m \)

is onto. For simplicity, we use the diagram.
The Whitehead Theorem: If $X$ is a CW-complex and $\epsilon: Y \to Z$ an $n$-equivalence ($n \in \mathbb{N}$ or $\infty$), then

$$[X, \epsilon]: [X, Y] \to [X, Z]$$
is a bijection when \( \dim X < m \) and onto
when \( \dim X \leq m < \infty \), and a bijection
for all CW complexes \( X \) when \( n = \infty \).

\[
T \Theta U^m \cong M
\]

\[
\text{dim } \Pi = M
\]

\[
M^m \wedge M^m = (M \times M)^{T \times 0^n}
\]

\[
\text{"diagonal"}
\]

\[
\Sigma^m M^\infty
\]

\[
? = \Sigma^m M^\infty \wedge \Pi^n
\]