Grothendieck topology (on a category \( \mathcal{C} \) with pullbacks) is given by a system of covers:

1. An \( \text{morphism} \quad y \xrightarrow{\sim} x \) in \( \mathcal{C} \).

A set of morphisms with the same target \( y \rightarrow x \).

2. Transitivity: If \( \{ X_\alpha \rightarrow X \} \) and \( \{ X_\beta \rightarrow X_\alpha \} \) are covers, then \( \{ X_\beta \rightarrow X \} \) is also a cover.
3) If \( \{X_\alpha \to X\} \) is a cover and \( f : Y \to X \) any morphism then \( \{X_\alpha \times_Y Y \to Y\} \) is a cover.

\[
\begin{array}{ccc}
X_\alpha \times_Y Y & \to & X_\alpha \\
\downarrow & & \downarrow \\
Y & \to & X
\end{array}
\]

\text{pullback}

\text{notation}

\text{Examples: 1) } \mathbb{R} \text{ a topological space,}
\[ \mathcal{G} \text{ is the category } \text{ of } \text{ open sets in } X \]

\[ U \to V \quad \text{if } U \subseteq V \]

Exactly one morphism when \( U, V \) open sets. Covers are set of inclusions of open sets \( \{ U_\alpha \subseteq U \} \) such that \( U \cup \bigcup \alpha U_\alpha = U \).

(pullback = intersection)

2. Smooth schemes over Spec \( k \) where \( k \) is a field.

(disjoint unions of smooth quasi-projective varieties over quasi-affine \( k \))
Zariski topology (closed subset an intersection of affine \textit{vs.} projective algebraic sets with \( X \)).

\[ X \subseteq k^n \text{ \emph{vs.} } X \subseteq P^n. \]

basically, this is an example of type 1.

\underline{A Modification:} "We don't modify the covers, but the category."

\[ \mathcal{Y} = \text{category of all smooth schemes / Spec } k \]

\[ \lceil \text{II quasi-affine varieties} \rceil \]

\[ \lceil \text{all II quasi-proj. varieties} \rceil \]
morphisms \leq \text{ continuous maps, preserve regular functions}

In a neighborhood of a point \( x \),

\[
\frac{f}{g} \quad \text{poly}
\]

\( g(y) \neq 0 \) for \( y \in U \).

((in quasi-projective case, homogeneous of the same degree))

\[
\psi : X \rightarrow Y
\]

\[
\psi(x) = y
\]

\[
\forall v \in Y \quad \exists u \in X \quad \psi(u) = v \quad \forall v \in Y
\]
1. Zariski open: regular function \( f : U \to k \)
   for \( f \) is regular.

Covers: \( \{ U_\alpha \to U \} \quad U_\alpha \subseteq U \)

Tenderi: open subset of \( U \)

\( \bigcup U_\alpha = U \).

3. Étale topology: \( \mathcal{G} \) = smooth scheme over \( k \)

Covers:
\[ \{ U_\alpha \to U \} \]

systems of étale morphisms
\[ \Pi U_2 \to U \]
onto closed points (over $k_{sep}$)

4. Nisnevich topology

covers
\[
\{ U_2 \to U \} \text{ are étale covers}
\]

Additional condition: For every point $x$ of $U$
defined over a field $L \supset k$,

there is a point $y$ of some $U_2$, $y \to x$ a morphism
and \( y \) is defined over \( L \).

**Example:** \( k = \mathbb{Q} \), \( x^2 : \mathbb{Q}[\alpha] \to \mathbb{Q} \) is an étale cover, not a \( \mathbb{Q} \)-cover.

\[
(\mathbb{Q}[\alpha]) = \mathbb{Q}[x, x^{-1}]
\]


look at a point \( a \in \mathbb{Q} \setminus \mathbb{Q}^2 \)

\[
\text{the point over } a \text{ in } \mathbb{Q}[\sqrt{a}],
\]
defined over \( \mathbb{Q}[\sqrt{a}] \).

Sheaves in a Grothendieck topology.
\[\{\text{set of all groups, rings, etc. similarly}\}\]

\[\text{are functors (contravariant on } C)\]

\[F : C^\text{op} \to \text{set}\]

For a cover \(\{X_\alpha \to X\}\), first index

\[F(X) \to \prod F(X_\alpha) \to \prod_{\alpha, \beta} F(X_\alpha \times_{X} X_\beta)\]

\[\uparrow \quad \alpha, \beta\]

Second index.

\[\text{is an equaliser.} \quad \{ f : \bigwedge_{\alpha} \to \bigvee_{\beta} \}\]
\[ \text{s} \rightarrow x \]

\[ f_\alpha = g_\alpha \]

Example: In the Zariski topology, the sheaf of regular functions.

Pre-sheaf is simply a functor

\[ F : \mathcal{O}_\mathfrak{X} \rightarrow \text{Sets} \]

We can also think of the category of pre-sheaves (isomorphisms are natural transformations), and the full subcategory of sheaves. The category of pre-sheaves has limits and colimits (“point-wise”).
Forgetful functor: $\text{Sheaves on } \mathcal{E} \rightarrow \text{PreSheaves on } \mathcal{E}$

with sheafified topology

It has a left adjoint $L$ ("sheafification").

We can make limits of sheaves by working object-wise, make colimits by forming colimit in the category of pre-sheaves and then sheafifying.

Then applying $L$

The main reason we are interested in sheaves
In Voevodsky’s setup is a a kind of a “relaxation” of objects.

How “an object a sheaf”?

An object always determines a pre-sheaf \( \varphi \) extendable to a sheaf

\[ \varphi \rightarrow \text{sets} \]

\[ X \in \text{Obj} \mathcal{E} \]

\[ \varphi (\cdot, X) : \varphi^{0h} \rightarrow \text{sets} \]

\[ \overline{\text{morphisms in } \mathcal{E} \cdot \varphi : \cdot \rightarrow X} \]

Is this pre-sheaf a sheaf?
A Grothendieck topology is called not-canonicial when every representable pre-sheaf is a sheaf.

[Reason for terminology: Canonical Grothendieck topology:]

\[ \text{when} \]

\[ \prod U_\alpha \times U_\beta \rightarrow \prod U_\alpha \rightarrow U \text{ is a co-}
\]

\[ \text{equaliser.} \]

Theorem: flat, Nisnevich, étale topologies are
Voevodsky's homotopy theory takes place in the category of Nisnevich sheaves on the site of smooth schemes $/$ $k$. We will insert the word "simplicial".

- simplicial objects
- model structures
- Examples: simplicial sets (algebraic topology as
a part of algebra

- 'simpler' vanishing sheaves \( \mathcal{F} \) on category of interest.