A brief primer on schemes

Affine schemes - from one point of view,

(commutative rings) \rightarrow R

\text{Spec}(R) \leftrightarrow R

(topological space = \{ prime ideals of } R \}\)

Taniyama topology: Closed set \mathbb{Z}(I) is determined

by an ideal I: \{ p \text{ prime ideals } | I \subseteq p \}
Why we talk about prime ideals (and not maximal):
Replacement for Nullstellensatz: In any comm. ring \( R \), for any ideal \( I \):
\[
\sqrt{I} = \bigcap p,
\]
\( p \in I \text{ prime} \)

(Exercise: Prove it or look up the proof.)

The actual Nullstellensatz: If \( R \) is a finitely generated algebra over a field, then
\[
\sqrt{I} = \bigcap m
\]
"Goldman ring" \( \mathfrak{m} \supseteq I, \mathfrak{m} \text{ maximal} \)
Grothendieck's view of geometry:

A Ringed Space: A space $X$ with a sheaf of rings $\mathcal{O}_X$.

The structure sheaf $\mathcal{O}_X$ is a continuous map,

Morphisms $X \rightarrow Y$, $\mathcal{O}_Y \rightarrow f^* \mathcal{O}_X$,

$$(f^* \mathcal{O}_X)(u) = \mathcal{O}_X(f^{-1}(u))$$

for open sets $u$ in $Y$. 
Locally ringed space $X$: $x \in X$

$\mathcal{O}_{X,x} = \varprojlim_{U \ni x} \mathcal{O}_X(U)$ is a local ring (has a unique maximal ideal).

For morphisms: For every point $x \in X$, $\varphi: \mathcal{O}_Y \rightarrow \mathcal{O}_X$, there is an equivalent maximal ideal of the ring $\varphi^{-1}(m_{x,x}) = m_{y,y}(x)$ maps into the induced map from the morphism of sheaves.
Example: \( \mathbb{A}^1_k = \text{Spec } k[x] \)

Local ring at 0: \( (k[x])_{(x)} \overset{\rho(x)}{\rightarrow} q(x) \)

\( q(x) \text{ not a multiple of } x \)

Not allowed:

\( (k[x])_{(x)} \overset{\rightarrow}{\rightarrow} k(x) \)

\( \uparrow \)

unique

maximal ideal is \((0)\)

maximal ideal is \((x)\)

\( 0 \text{ closed point} \)
"$\mathbb{A}^1 = \text{Spec } \mathbb{C}[x]"$

$\mathbb{A}^n \cong \text{Spec } \mathbb{C}[x_1, \ldots, x_n]$

$(0)$ is a point of $\mathbb{A}^1$, called the generic point of $\mathbb{A}^1$.

The generic point $(0)$ is in $\mathbb{A}^1 \setminus \{0\}$, not in $\{0\}$.

**Definition:** A scheme is a locally ringed space $X$ such that every $x \in X$ has an open neighborhood $U$ isomorphic (as a locally ringed space) to an affine scheme.
Locally ringed spaces have arbitrary limits and colimits, schemes do not.

Affine schemes do have limits and colimits (because commutative rings do).

The definition of smooth morphisms I gave before works on schemes. (Affine neighborhoods in a scheme form a basis of topology. Exercise.)
\[ \text{Spec} R \text{ in the category of schemes} \]
\[ \text{exist, it is a discrete space} \]
\[ \text{in the Zariski topology, it is smooth / smooth, it is not affine.} \]
\[ \text{is a right adjoint} \]
\[ \text{Affine schemes} \subseteq \text{Schemes} \text{ preserves limits} \]
\[ \text{Spec } \prod_{N} \text{ (X)}, \quad \presuperscript{\text{Spec} (X)}{\text{Spec} (\prod_{N} k)}{\text{?}} \]
\[ \text{Why} \quad \prod_{N} \text{Spec} (k) \neq \text{Spec} \left( \prod_{N} k \right) ? \]
The Hasse diagram for the Boolean algebra is:

\( \text{Convergent chain condition)} \)

... for some \( m \).

This has to be

a compactification (Stone-Čech compactification)

\( \text{ultrafilters,} \)

\( X \uparrow \leq 2 \quad y \leq 2 \quad z \leq 2 \quad - \quad X \downarrow \)

\( \text{Descent chain condition)} \)

for closed subschemes.
Example: \[ \text{If } \text{Spec}(k) \text{ is not Noetherian,} \]

To define Nisnevich topology, we can take:
- Smooth schemes / Spec \( k \)
- \( \Pi \) smooth quasi-projective varieties / Spec \( k \)
- \( \Pi \) quasi-projective

or add a \underline{Noetherian} condition.

A Nisnevich cover (in any of these categories)
is an étale cover \( X_\alpha \to X \) such that for every point \( x \in X \) (not necessarily closed), there exists a point \( x_\alpha \in X_\alpha \) \( \text{ s.t. } f_\alpha(x_\alpha) = x \) with \( \text{a morphism } \alpha \).\[\begin{array}{c}
\mathcal{O}_{X_\alpha, x_\alpha} / \mathcal{O}_{X, x} \\
\cong \mathcal{O}_{X_\alpha, x_\alpha} / \mathcal{O}_{X, x}
\end{array}\] (more precisely, \( \text{the canonical map is an isomorphism} \).)

A Néron model square:
Theorem: If we work in the category of Noetherian smooth schemes over $\text{Spec}(k)$, then a presheaf $\mathcal{V}$ is a Nisnevich sheaf if and only if it takes every Nisnevich square to a pullback.
\[ f(u^x) \leq f(u), \]

\[ f(u) \leq f(x). \]

**Proof (next time).**