Quillen model category (homotopy theory in an abstract context)

A category $C$ with subcategories

\[ \rightarrow \] cofibrations

\[ \rightarrow \] fibrations

\[ \sim \] equivalence

1. $C$ has limit and colimit of finite diagrams

2. two out of three are equivalences, so is the third
3. \( \rightarrow \), \( \sim \) are preserved by \( \text{det} \):

\[
\begin{array}{c}
\text{Id} \\
\downarrow
\end{array}
\begin{array}{c}
\sim \text{Id} \\
\downarrow
\end{array}
\begin{array}{c}
a \rightarrow b \\
h \rightarrow \text{Id}
\end{array}
\]

\( a \) is a retract of \( b \)

Exercise:

- Lifting of an epimorphism is an epimorphism

4. \( \sim \)

\[
\begin{array}{c}
\text{LCP} \\
\downarrow
\end{array}
\begin{array}{c}
\sim \\
\downarrow
\end{array}
\begin{array}{c}
\text{RCP} \\
\downarrow
\end{array}
\]

\( \sim \) : acyclic cofibrations

\( \text{acyclic fibrations} \)

acyclic fibrations are precisely those morphisms which have LCP w.r.t. to all fibrations

acyclic fibrations are precisely those morphisms which have RCP w.r.t. to all fibrations
Examples:

1. $C = \text{topological spaces}$
   \[\rightarrow\] cofibrations (see May)
   \[\rightarrow\] (Hurewicz) fibrations
   \[\sim\] homotopy equivalences

   Strong model structure

2. 695: 
   $C = \text{topological spaces}$
   \[\sim\] weak equivalences
   \[\rightarrow\] retracts of CW-pairs
$X \xrightarrow{a} Y \xrightarrow{h} Z \xrightarrow{\text{Id}} Z \xrightarrow{(z, X)} \times \text{a CW-pair}$

$\xrightarrow{\text{RLP with respect to}}$

$\text{CW-pairs } X \rightarrow Y \text{ which are equivalences}$

$\xrightarrow{\text{RLP with respect to}}$

$D^n \xrightarrow{\text{in Hurewicz function, added of } 0^n, \text{my name}}$
Example: Covering spaces are fiber filtrations not necessarily Hausdorff filtrations.

3. Simplicial sets

\( \Delta = \Delta^0 \rightarrow \text{set} \)

\( \text{Obj} \Delta = N_0 \)

\( \Delta(m, n) = \text{non-decreasing maps} \)

\( m \rightarrow \{0_1, \ldots, m\} \subseteq \{0_1, \ldots, n\} \)
Cofibrations: \( S \to T \) such that 
\( S_a \to T_a \) is injective for each \( a \)

Fibrations: Kan fibrations

\[ \Delta^n \]

(simplexwise set: unsorted functions 
\( (\Delta^n)_m = \Delta(m, m) \))

Simplicial complex (with ordered vertices) 
\( \to \) simplicial set.
\[ V(m, k) = \Delta_m \setminus \{n - \mathbf{k}\} \]

dimensional simplex,
and the \(k\)th face
\[ k = 0, \ldots, m \]

\[ V(m, k) \hookrightarrow \Delta_m \]

can fibrations are
maps of simplicial
sets which satisfy
RLP\ w.r.t. to \(\otimes\).

\[ V(m, k) \rightarrow E \]
\[ \Delta_m \rightarrow B \]
This determines equivalences.

In general, cofibrations and fibrations determine equivalences.

First, acyclic cofibrations are precisely those morphisms satisfying LLP under fibrations.

Similarly, acyclic fibrations are precisely those morphisms which satisfy the LLP under all fibrations.

Well, strictly, an equivalence is a map of simplicial sets which, after topological realisation, becomes a weak equivalence $\Rightarrow$ a homotopy equivalence.
To determine if a morphism is an equivalence, factor it as \( i \circ j \circ f \).

\[
\begin{array}{c}
\xrightarrow{i} \\
\xrightarrow{j} \\
\xrightarrow{f}
\end{array}
\]

\[ i \text{ is an equivalence } \iff \text{only if } j \text{ is an equivalence } \]

Exercise: Also show that \( (\circ) \) compositions & equivalences
in any model structure \(\Rightarrow\) determine fibrations,

\(\circ\) fibrations & equivalences determine cofibrations.

Future: two model structures on simplicial Nisnevich sheaves

one simplicial \(\Rightarrow\) on any kind of simplicial sheaves

\(\text{free: } A^1 \to \ast\) to be an equivalence

(\(\text{localization of model categories}\) )
What benefit does a model structure give us?

Quillen:

- A derived category (homotopy category)

We also have a systematic treatment of derived functors.

\[ F \quad C \leftarrow D \]

\[ G \quad F \text{ left adjoint to } G \]

\[ C/D \text{ have model structure} \]
We say that an object $X$ is cofibrant if:

1. $X$ is a cofibration.
2. $F$ preserves cofibrations.
3. $C$ preserves cofibrations.
4. $C$ preserves fibrations and acyclic fibrations.
5. $F$ preserves cofibrations and acyclic cofibrations.
$X$ is terminal if $X \rightarrow * \in A$ is a terminal object.

**Theorem:** A Quillen adjunction determines an adjunction on derived categories "by applying $F$ to leftward objects and $C$ to rightward objects." desired category

$DC \xrightarrow{LF} DD$ R.G.

These are, by definition, the derived functors. $\square$
How to get the derived category of a model structure:

1. Replacing object by cofibrant (fibrant) object.

\[
\emptyset \rightarrow X' \sim X \rightarrow X'' \sim X
\]

Now $X'$ is cofibrant.

Now $X''$ is cofibrant.
We have a notion of homotopy of two morphisms. (In fact, left homotopy and right homotopy.)

with fibrant-cofibrant objects, these are equivalent.

Theorem: $D C = \{ \text{fibrant-cofibrant objects, homotopy classes of morphisms} \}$

Details next time.