Bousfield localization of the simplicial model structure on simplicial presheaves (sketched in Nolev-Veredovsky detail: Jardine: Local Homotopy Theory, 2014)

\[ A = \text{a set of morphisms} \]
\[ \text{of simplicial presheaves on a small site } T \]

In the category \( \Delta^h \text{-Presh}(T) \) (simplicial presheaves)

\[ X \in A \text{-local} \text{ if } f \in A \text{ and } f \not\in Y \]
\[ D \Delta^0 - \text{Push} (\tau) (Y \times \mathbb{Z}_2, X) \xrightarrow{\sim} D \Delta^0 - \text{Push} (\tau) (Y \times \mathbb{Z}_2, X) \]

derived

(= homotopy category as a bijection.

An $A$-weak equivalence is a morphism $f \colon X_1 \rightarrow X_2$ such that $D \Delta^0 - \text{Push} (f, i) \colon D \Delta^0 - \text{Push} (X_2, Y) \rightarrow D \Delta^0 - \text{Push} (X_1, Y)$

induced by $f$ is a bijection for every $Y$ local.

Theorem: Simplexwise cofibrations, $A$-weak equivalences and $A$-fibrations (KLP with respect to simplexwise
(of functors which are $A$-equivalences) form a model structure.

**Motivation:** This can be done in any category $C$ with equivalences, some not $A$. The idea is to force the element of $A$ to also be equivalences.

Typically (also in our case), we construct a localization:

\[ X \rightarrow L_A X \] where $L_A X$ is $A$-local

$A$-equivalence
Exercise: If $X_1, X_2$ are $A$-local then an $A$-equivalence $f: X_1 \to X_2$ is an equivalence.

If this happens (as we will prove, it does for $D^{[\mathbf{GrSh}(T))]}$) then $D_A C$ is the derived category.
of C with respect to $A'$-equivalences.

To prove the theorem, the key point is to show that there is a set $B'$ of cofibration $A'$-equivalences (as opposed to class)

such that $X \rightarrow C$ is an $A'$-fibration iff it has RLP with respect to the element of $B'$

(then attaching $B'$-elements via pushout play the role of "a $\alpha$-dyne extension").

Let $A'$ be a set of cofibration representatives of $A$. 
Proposition: A morphism $X \to Y$ is an $A$-equivalence ($= A$-weak equivalence) iff for every simplicially fibrant $A$-local $Z$,

$$\text{Map}(Y, Z) \to \text{Map}(X, Z)$$

is a simplicially weak equivalence. $\square$

Prop. If $Z$ is simplicially fibrant, $A$-local then so is $\text{Map}(X, Z)$ for every simplicial sheaf $X$. 
\((x, \text{ which is the adjoint, is symmetric})\). □

Lemma: \( X \xrightarrow{a} Y \)

\[ \begin{array}{c}
\text{b} \\
\downarrow \\
X' \xrightarrow{c} Y'
\end{array} \]

If \( b \) is an \( A \)-equivalence, so is \( d \).

If \( a \) is an \( A \)-equivalence, so is \( c \). □

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"Building cells for \( A \)-localisation"

(Still a little weaker than what I look for testing \( A \)-fibrations).
Let $U \in \text{Obj } T$.

Let $\beta_i = \{ u_i, f_i \}$. 

Proposition: If $X$ is weakly locally fibrant, then $X$ is $A$-local if and only if $X \to \ast$ has RLY with
Proof: The RLP mentioned occurs if and only if for every \( U \in T \) and any \( f: \mathbb{R} \to X \in T \) the morphism of simplicial sets

\[
\Delta^0 \times U \times \Delta^n (U < U', X) \to \Delta^0 \times \text{Re}_{X} (U \times U', X)
\]

has the RLP with respect to embeddings \( \partial^n \to \Delta^n \) (i.e. is an acyclic fibration of the floral set).

Now use Propositions 1 & 2. \( \square \)

Corollary: There exist a set (as opposed to a class)
$B$ of cofibrations an $A$-weak equivalences such that for a simplicial presheaf $X$, if $X \to \mathcal{B}$ has the RLP with respect to $B$, then $X$ is $A$-local.

Proof: When we constructed the simplicial model structure on simplicial presheaves, we proved that there exist a set $\mathcal{B}_0$ of cofibration equivalences such that a morphism is a simplicial cofibration if and only if it satisfies RLP with respect to $\mathcal{B}_0$ (cardinality bound dependent on $T$ which is small).

Put $\mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_1$. $\square$
Now we can construct

$$X \rightarrow \Delta_X$$

\(\Rightarrow\) \(A\)-local (firmest, if you
\(A\)-equivalence like)

by attaching, transfinitely, via preprints, elements
in \(B\). Eventually, by sheer cardinality, you have a
small object as you need.

The magic lemma for constructing the \(\tilde{B}'\)
(which gives us the "amodyne extension") is
Lemma: suppose we have a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow & & \\
Z & \xrightarrow{j} & Y
\end{array}
\]

such that \( i \) is an \( A \)-equivalence and \( Z \) is \( i \)-bounded (cardinalities of all sections are \( \leq K \)).

Then there exists \( X_0 \xrightarrow{i_1} X \)

\[
\begin{array}{ccc}
X_0 & \xrightarrow{i_1} & X \\
\downarrow & & \\
Z & \xrightarrow{j} & Y
\end{array}
\]

\( i' \) \( k \)-bounded such that \( i' \) is an \( A \)-equivalence.
Finish next time.