The simplicial realization of a simplicial presheaf

\[ \Delta = \{ (x_0, \ldots, x_n) \in \mathbb{A}_{\mathbb{R}}^{n+1} \mid \sum x_i = 1 \} \]

\[ (= \text{Spec } \mathbb{R}[x_0, \ldots, x_n] / (x_0 + \ldots + x_n - 1) ) \]

If \( X \) is a simplicial (pre) sheaf, more generally, object of another category,

\[ |X| = \Delta \times_{\Delta^0} X \]

\[ \mathcal{C}(\ast, \Delta_m \times X_n) \Rightarrow \prod_{m \geq n} \Delta_m \times X_n \]
Theorem: \( \{X\} \sim X \).

The \( \mathcal{A} \)-sided bar construction of categories.

\[ \beta(\{X, C, Y\} \rightarrow X \times C^Y) \]

\[ \uparrow \]

Simplified: with stages: \( a_0 \rightarrow f_1 \rightarrow f_2 \rightarrow \ldots \rightarrow f_n \rightarrow a_m \)

on the left:

contravariant \(-\).

on the right:

functions \( Y: C \rightarrow D \)

Exercise: check \( \square \)
This means pullback

\( \forall \times \text{Obj} C \to \text{Obj} D \to \text{Obj} E \to \text{Obj} F \to \text{Obj} G \)

(sorry)

faces: functoriality \( \times \text{fun} \)

degeneracies: Insert Id.

composition \( \text{fun} \)

key point: face maps are cofibrations

\( \Delta \times \Delta^0 \to X \)

\( \Rightarrow \nabla \leftarrow \beta (\Delta, \Delta^0, X) \to \beta (\Delta^0, \Delta^0, X) \leftarrow \beta (\Delta, \Delta^0, X) \)

\[ \delta m = \delta^0 (m, m) \]

reprentable:

\[ \beta (+, \Delta^0, X) \leftarrow \beta (\Delta^0, \Delta^0, X) \]

\( \nabla \leftarrow \beta (\Delta, \Delta^0, X) \)
\( \mathbb{D}(0, \sigma_0, 0, \sigma_k) \Rightarrow X \)  

Differential  

\[ \Delta + \sigma_0 X = X \]

Twist in algebraic geometry

Voevodsky: \( \mathbb{P}^1 \sim S^1 \times \mathbb{C}^\times \)

\( S^1 \sim \mathbb{A}^1 / 0 \sim \)

Exercise.

\( C_m = S^{1,1} \sim S^2 \)

\( \mathbb{P}^m / \mathbb{P}^{m-1} \sim \bigwedge \mathbb{P}^1 \sim \bigwedge S^{1,0} \times \bigwedge S^{1,1} \)  

"Twisted line"
Example of twist: Basic Hodge theory, geometry.

Let $X$ be a complex manifold. ("locally, smooth")

We will assume $X$ is compact, in good cases Kähler:

The de Rham complex on a complex manifold $M$:

$$ TM \xleftarrow{\text{tangent bundle of } M} TM^* \xrightarrow{\text{Hom} \left( TM_x, \mathbb{R} \right)} $$

We will complexify again: $ TM_x^* \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\text{Hom} \left( TM_x, \mathbb{C} \right)} $
If \( z_1, \ldots, z_m : U \rightarrow \mathbb{C} \) are holomorphic coordinates: \( z_j = x_j + i y_j \).

A basis of \( T_{\mathbb{C}} U \otimes \mathbb{C} \):

\[
\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \quad \text{and} \quad \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_j}
\]

\[
\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \quad \frac{\partial}{\partial \overline{z}_j} = 0
\]

\[
\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).
\]
$d_{\bar{z}_j}, d_{\bar{z}_j}$ are the dual basis of $\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}$.

Now the de Rham complex of $\Omega$ is double-graded:

$$L^{k, \ell}(\Omega) = \{ \sum_{i_1 + \cdots + i_p = \ell} \sum_{j_1 + \cdots + j_p = k} a_{i_1 \cdots i_p j_1 \cdots j_p} \text{d}z_{i_1} \wedge \cdots \wedge \text{d}z_{i_p} \wedge \text{d}\bar{z}_{j_1} \wedge \cdots \wedge \text{d}\bar{z}_{j_p} \}$$

$C^\infty \text{ smooth function } \Omega \to \mathbb{C}$

Similarly, $L^{k, \ell}(M)$ (in local coordinates).

$\gamma : L^k(M) \to L^{k+1}(M)$
In local coordinates,

\[ \partial \quad \partial \quad d z_j \quad \cdots \quad \land d \bar{z}_k. \]

\[ = \sum \frac{\partial h}{\partial \bar{z}_k} \quad d\bar{z}_k \land d z_j \land \cdots \land d \bar{z}_k. \]

\[ \partial h (z_j, \cdots, d \bar{z}_k) = \]

\[ = \sum \frac{\partial h}{\partial \bar{z}_k} \quad d\bar{z}_k \land d z_j \land \cdots \land d \bar{z}_k. \]

De Rham cohomology (with coefficients in \( C \)) is the cohomology of the total complex.
There is an operator of complex conjugation of differential forms:
\[ dz = d \bar{z} \]
\[ \bar{z} \text{ in coordinate } \bar{z} \text{ (ordinary complex conjugation)} \]
\[ (\text{we check rule to show this is preserved by holomorphic change of coordinates}) \]
\[ (\text{smooth version}) \]
A Hermitian metric is an Hermitian metric on a C*-module (complex inner product)
Example: smooth projective variety $\mathbb{C}^n$. The manifold $\mathbb{C}^n$ is called a \textbf{K"{a}hler} manifold if $\omega = 0$.

$\omega = \frac{1}{2} (\overline{\alpha - \beta}) e_\omega$

$L_{\alpha} : \text{real flat "gradient"}$

$\alpha^* = e_{\alpha} \in \mathbb{R}^n$
Fubini–Study metric

\[ w = i \partial \overline{\partial} \ln \| z \|^2 \]

and the homogeneous coordinates.

If \( M \) is a smooth projective variety over \( \mathbb{C} \), this induces a Kähler metric on \( M \).

\[ \Omega (M) \]

\[ \tau \]

\[ \xi \]

\[ \eta \]

\[ \theta \]

\[ \xi' \]

\[ \eta' \]

\[ \theta' \]
$F^p \mathcal{L}(\mathcal{M}) = \bigoplus L^{r-2} \mathcal{M} \quad \text{for} \quad p' \geq p$

(de Rham)

Theorem: For compact Kähler manifolds $\mathcal{M}$, this $E'$ collapses to $E'$. $\mathcal{D}$

$F$ - cohomology is the cohomology!