Algebraic $K$-theory

$m$-Bundles on an algebraic variety $X/k$, ($k$ field)

$\tilde{H}^1_{\text{alg.}}(X, GL_n(k))$

\[ \varphi_{ij} : U_i \cap U_j \to GL_n(k) \]
morphism of varieties over $k$

(And: continuous, locally composed
with a regular function produces
a regular function)

usual discussion of cycle: $\gamma_{ij}.\gamma_{jk} = \gamma_{ik}$

\text{corollary} (we lost time)

For schemes, this is not intrinsic enough,
(because "regular functions" on a scheme are
not really functions).
locally = Spec \mathbb{R}

points = prime \ yz
ideals

But we have the structure sheaf \mathcal{O}_X of regular
functions. Intuitively, algebraic bundles are
just finitely generated locally free sheaves of
\mathcal{O}_X-modules.

If \( X = \text{Spec } \mathbb{R} \) (affine scheme):

finitely generated projective \( \mathbb{R} \)-modules.

\( K^0(X) = \{ \mathcal{L} \} \) classes of locally free vector bundles.
How do we define higher $K$-groups?

(Quillen)

An introduction on infinite loop space theory

(no known analogue for $A^1$-homotopy theory)

(Adams: Infinite loop spaces)

An infinite loop space is a generalized cohomology theory

$$E_m \rightarrow \Omega E_{m+1} \quad m \in \mathbb{N}_0$$

Each $E_m$ is called an infinite loop space.
Maybe a space \( Z \) is given. Would I be able to recognize that it is an infinite loop space?

\[ \Sigma Y \]

\[ \rightarrow \] has a multiplication which is associative, up to homotopy

\[ \Sigma Y \times \Sigma Y \to \Sigma Y \] concatenation of loops
The can be made strictly one-to-one, untwisted by varying length of the loop.
allow 0 length for constant loop
more loops

If $t = \pi X$ (double loop space)
this composition is commutative
up to homotopy

shade the area $\sim \pi$
An infinite loop space has a binary operation which is commutative, associative, up to all possible higher homotopies.

What does this mean rigorously?

- Boardman - Vogt
- Waldhausen
- May - (Thomason)
- Stasheff
An operad is a formalism that encodes algebraic operations on multiple variables, every variable of an n-ary operation must occur precisely once: \( \cdot \) (in spaces)

**Definition:** An operad consists of a sequence of spaces \( \mathcal{O}(n) \), \( n = 0, 1, 2, \ldots \) \( \Sigma_n \) (symmetric group)

action \( \gamma(n) \), \( I \in \mathcal{O}(1) \) unit, composition:

\[
\gamma : \mathcal{O}(n) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_m) \to \mathcal{O}(k_1 + \cdots + k_m)
\]
Axioms: Unitality

\[ \varphi(n) \xrightarrow{\text{Id} \times 1^n} \varphi(n) = \varphi(1) \ldots + \varphi(1) \]

\[ \varphi(n) \rightarrow \varphi(1) \times \varphi(n) \]

Associativity of composition:
\[ Y^{(n)} \times D(k_1) \times \ldots \times D(k_m) \times D(k_{l_1}) \times \ldots \times (l_{i_1})^n \]

\[ \ldots Y(l_{m}) \times \ldots Y(l_{m+k_m}) \]

\[ \downarrow \]

\[ Y(k_1 + \ldots + k_m) \times Y(l_{i,j}) \]

\[ \ldots \times Y(\sum l_{i,j}) \]

Equivariance: (with respect to)
$\mathfrak{g}$-space $X$: Operations $\Theta: \mathfrak{g}(\mathfrak{g}) \times X \to X$

Cartesian product

(The endomorphism space: $\text{End}(X)(\mathfrak{g}) = \text{Map}(X^\mathfrak{g}, X)$)

A $\mathfrak{g}$-space is given by a morphism of spaces $\mathfrak{g} \to \text{End}(X)$
An $E_\infty$-operad is an operad $O$ where $O(n) \simeq \ast$.

We assume $X \simeq$ CW complex

$E_n$ acts freely.

An $E_\infty$-space is a space $X$ on which there acts an $E_\infty$-operad.
Theorem: 1. A connected \(E_{\infty}\)-space "is" an infinite loop space (also an \(E_{\infty}\)-space). If \(\pi_0 X\) is a group, it is necessarily abelian.

2. The interesting thing is when this assumption fails. If \(X\) is an \(E_{\infty}\)-space, there is a natural infinite loop space \(\text{gr}(X)\) and a map of \(E_{\infty}\)-spaces

\[ f: X \rightarrow \text{gr}(X) \]

such that via \(f\), \(H_*\text{gr}(X)) = H_*\text{gr}(X) [\pi_0 X^{-1}]\).
Example (of operads): The \( k \)-dimensional little cube operad

\[
\mathcal{E}_k(n) = I^k 
\]

\[
\mathcal{E} = \left\{ 0, 1 \right\}^k
\]

\[\Psi : I^k \to I^k \text{ linear (increasing embeddings with disjoint images)}\]

Lemma:

A \( k \)-fold loop space is a \( \mathcal{E}_k \)-space.
Proof: Use the order to compare loops.

\[ \mathcal{E}_0 = \bigcup \mathcal{E}_k \]

\[ \mathcal{E}_k \subseteq \mathcal{E}_{k+1} \]

\[ I \times [0,1] \]

Lemma: \( \mathcal{E}_0 \cap * \).

Corollary: An \( \infty \)-loop space is contractible.
$E_\infty$ yea! (via the graded $E_\infty$). □