\[ \text{Openad} \quad (\mathcal{E}(n))_{n \in \mathbb{N}_0} \]

(in spaces)

\[ \text{Space} \]

(1) \( \sum_{n} a_n \in \mathcal{E}(n) \)

(2) \( a \in \mathcal{E}(1) \)

(3) \( \psi : \mathcal{E}(n) \times \mathcal{E}(k_1) \times \cdots \times \mathcal{E}(k_m) \rightarrow \mathcal{E}(k_1 + \cdots + k_m) \)

Think: \( \mathcal{E}(n) = n \)-ary operations in \( n \) distinct variables.

Based: \( * \in \mathcal{E}(0) \).
The group $G$ acts on a space $X$: $G^m \times X \times \cdots \times X \rightarrow X$.

Actions:

- Action: $\cdot : \ast \in G \rightarrow \ast \ast X$.
- (X bound)

A monad in a category $T$: $C: T \rightarrow T$.

- $\mu : C \Rightarrow C$ associativity,
- $\eta : 1 \Rightarrow C$ unit law.

$C$-algebra $X$: $\Theta : CX \rightarrow X$ associativity, unit law.

A monad associated with a group $G$: 

Unbundled: \[ CX = \bigoplus_{m \geq 0} \Sigma_m \cdot X^m \]

Based: \[ CX = \bigoplus_{m \geq 0} \Sigma_m \cdot X^m / \sim \]

\[ \delta_i : \Sigma_m \to \Sigma_{m-1} \quad (\text{plug in last point}) \]

\[ \delta_i : X^{n-1} \to X^n \]

Observation: A \( C \)-algebra is the same thing as a \( G \)-algebra.
The principle of loop space theory

\[ \xi_k : G_k(\tau) = \]

little k-cube operad

The associated monad \( C_k \) has a monad of monads

\[ C_k \rightarrow S^k C_k \]

(base)

\[ \rightarrow k\text{-fold suspension} \]

\[ \rightarrow k\text{-fold loop} \]
\[ C_k X = \{ x \in X \} \quad \text{and} \quad k \geq 2 \]

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Approximation theorem: If \( X \) is connected (\( W \)-complex):

\[ C_k X \rightarrow \Omega S^k E^k X \quad (1) \]

is a weak equivalence. Otherwise, (1) is a group completion.

\[ H_\ast \Delta^k E^k X = H_\ast C_k X \mathbb[Z \pi_0 C_k X] \]

\[ \square \]
Recognition principle - how to recognize a k-fold loop

Suppose \( X \) is a \( C_k \)-space. \( \mathcal{E} \rightarrow \mathcal{E} \)

\[ B(\mathcal{E}^k, C_k, X) \]

\[ \mathcal{E}^k \leftarrow \mathcal{E}^k \text{- function} \]

\[ (\mathcal{E}^k \circ \mathcal{E}^k \rightarrow \mathcal{E}^k) \]

\[ E \text{ right } C \text{- functor} \]

\[ D \text{ left } C \text{- functor} \]

\[ C \circ D \rightarrow D \]

\[ B(E, C, D) \]

\[ \wedge \text{ - welded} \]

\[ m \text{- the stage} \]

\[ E \circ D \]

\[ \text{face - composition} \]

\[ \text{degenerate, unit} \]

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May: A n equivalence. Quasi-fibrations (Dold, Thom: Quasi-
\[ B(C_k, C_k) \xrightarrow{\sim} S^k B(C_k, C_k) \]

\[ \text{an equivalence if } X \text{ connected, group completion otherwise} \]

\[ \{X, \partial X\} \xrightarrow{\sim} \text{ (hierarchical level-wise)} \]

Passing to \( k \to \infty \), \( E_\infty \) - space is (in group completion)

\[ \text{an infinite loop space, } E_\infty \]

\[ U \mathbb{C}_k \xrightarrow{\sim} \mathbb{Z}_m \xrightarrow{\partial} \mathbb{Z}_{m+1}, \ m = 0, 1, 2, \ldots \]

Homotopy class \( \sim [X, \mathbb{Z}_m] = E^\infty X \)
Any $G_0$ -space algebra

is a group - complete

map of $G_0$ - spaces $B(P, X)$ is an equivalent $G$ - algebra.

$(\text{free } G_0 \text{-action})$

$G_0 \text{-operad}: \phi_0^i (x) \in k$
A permutative category is a category $\mathcal{C}$ with an operation $\otimes$ which is strictly associative and unital, and I have a natural transformation $\sigma : X \otimes 1 \to 1 \otimes X$

1. $\sigma^2 = \text{Id}$
2. $X \otimes Y \otimes Z \xrightarrow{\sigma} Y \otimes X \otimes Z$

$(\eta \otimes 1)(\otimes X) = Y \otimes Z \otimes X$
Perhaps a more familiar notion: Symmetric monoidal category

\[(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)\]

\[0 \otimes X \cong X \cong X \otimes 0\]

Coherence diagrams commute.

( take a word in a free commutative monoid, process using axioms --- --- with the original word)

Street- Joyal construction: For every symmetric monoidal category there is a canonical equivalent permutative category.
From a permutative category (hence, a symmetric monoidal category),

we can manufacture an $\infty$-space (hence an $\omega$-loop space, hence a generalized cohomology theory).

The nerve (classifying space of a category):

$\mathcal{N}(J) \subset \text{ $n$-th stage: } \prod_j J \times \mathcal{O}_J \times \prod_j J \times \cdots \times \mathcal{O}_J \times \prod_j J$

$\omega$-tuples of composable morphisms,

($0$-tuple = object)
If $T$ is a permutative category, why is $BT$ an $E_{\infty}$-space? I produce a simplicial operad: tech resolution of the associative operad $\mathcal{C}(S)$.

\[ S \leftarrow S(m) = \Sigma m \]

An $S$-algebra is the same thing as a monoid.
\[ \tilde{C}(S)(n) = \underbrace{S \times \cdots \times S}_{(k+1) \text{ times}} \]

\[ \text{simplex level} \quad \uparrow \]

\[ \text{faces = projections} \]

\[ (\text{If } G \text{ is a group, } \tilde{C}(G) \cong B(G, G^+) ) \]

\[ \text{degeneracies = diagonal elements} \]

\[ \tilde{C}(S) \text{ is a simplicial operad acting on } D(T) \]

\[ \text{simplicially (one stage at a time)} \]
After simplification, mix them up. Use $\Theta$. 
Theorem: If \( J \) is a commutative category, then \( BT \) is naturally a \( \mathcal{D} \)-space if hence an \( E_\infty \)-space.

\[ E_\infty \text{-operad} \]

Examples:

1. \( J = \text{finite set} \), \( \sim \), \( H \)

2. \( S = S^0 \), the sphere spectrum

3. f.d. \( C \)-vector space, \( \sim \), \( \mathcal{D} \)
Type spectral category $k \in$ connective topology

$R$-

Projective f.g. $R$-modules $R$ ring

Locally free algebra bundles $X$ scheme

$\cong 1 \circ$

$\rightarrow$

$K_{alg} R$

$K_{alg} X$. 