For simplicity, let $k$ be a field (but a similar discussion applies in the case of rings).

Then every $k$-module is free. It follows from a symmetric monoidal category

$\mathcal{O} = \{ \text{finitely generated projective } k\text{-modules, } \cong, \oplus \}$

is equivalent to the category (sheaf for) $\mathcal{O}$:

Obj $(\mathcal{O}) = \{ 0, 1, 2, \ldots \}$

Hom $(m, n) = \emptyset$ if $m \neq n$.

This yields category.
\[ BO = BO_{\mathbb{N}} = \prod_{m \in \mathbb{N}} BGL_m(k) \]

\[ \mathcal{E}_0 - \text{space}. \]

\[ k \left( GL_n(k), 1 \right) \]

\[ \mathcal{E}(\mathbb{N}, 1) - \text{spaces} \]

last time, we constructed a group completion

\[ BO_{\mathbb{N}} = \prod_{m \in \mathbb{N}_0} BGL_m(k) \rightarrow K \]

\[ \text{infinite loop space,} \]

\[ \text{the associated generalized} \]

\[ \text{cohomology theory} \]

\[ \text{is} \quad K_{\text{alg}}(k). \]

\[ x : \mathbb{Z} \rightarrow K \]
Pick } x \in \beta GL_1(k) \text{ \\

holding } (\beta \sigma \rightarrow \beta \sigma \rightarrow \cdots) \text{ \\

} GL_\infty(k) = \bigcup GL_n(k) \text{ \\

All } \sigma \in GL_n(k) \rightarrow (A^\sigma \quad 0) \in GL_{n+1}(k) \text{ \\

discrete group } \beta GL_\infty(k) \text{ is itself a } K(n,1) \text{- space. } \\

there are no higher homotopy groups, not an equivalent.

A more accurate proof: } \pi_1 \beta GL_\infty(k) = C_\infty(k), \text{ Not abelian! }
\[ \pi_1(S^2 \times X) = [S^2, X] \]

This shows \( \eta \) cannot be an equivalence.

(Note: \( BH \mapsto k \times \mathbb{Z} \) cannot be an \( G \)-space!)

If it was, it would be group-like, so it would have to be an infinite loop space, and therefore have an abelian fundamental group.)
Quillen’s first construction of $X$ was based on his $\pi_1$ observation. Under some conditions, Quillen made a “$+$-construction” which takes any connected CW complex $[\pi_1 X, \pi_1 X]$ perfect (its commutator is the same thing), by attaching 2-cells and 3-cells, he constructed a space $X^+$ and a map

$$X \to X^+$$

which is an $\equiv$ in homology, abelianization in $\pi_1$. 
Here we deal with discrete groups, and apply $(\cdot^*)^+$ to get a topological space which has some higher homotopy.

Recall that in the motivic construction, there is no concept of a loop machine, we get directly an $A^1$-spectrum not even known!

with a loop space

\[ \Omega \times \mathbb{Z} \]

Different meaning from above!

\[ \Omega \mathcal{L}(k) = \mathcal{U}(k) \]

as algebraic groups!
\[ \text{Map}_{k}^{\Delta^1}(\mathcal{P}_{k}, \mathcal{B}L_{\infty}(k) \times \mathcal{Z}) \]

\[ \cong \mathcal{B}L_{\infty}(k) \times \mathcal{Z}. \]

in \text{\(A^1\)-homotopy category}

\[ \text{What is the relationship with Quillen's construction?} \]

\[ \varepsilon \text{ A1-homotopy paper.} \]

\[ \text{Theorem (Nerel-Voevodsky): For a smooth Noetherian separated } \]

\[ \text{algebraic group, } \]

\[ \{ \text{class of } m \text{-dimensional } \]

\[ \text{algebraic vector bundles on } \}

\[ \cong [X, \mathcal{B}GL_m(k)] \]
locally free rank r sheaves of $O_X$-modules

I will not give the proof, but will argue that it is a non-trivial theorem; it includes the fact that $\mathcal{E}$ classes of algebraic vector bundles on $X$,

\[ \{ -1 \} \] on $X \times M$. 

This was a conjecture of Grothendieck which was open until the 70's. It was solved by Quillen.

The property of a scheme as being mo
non-trivial f.g. projective vector bundles (modules)
is known as the Quillen–Suslin property.

Why don’t we need the $\pm$-construction when we use the $\mathbb{A}^1$-topology?

\[ \pi_1^{\text{f.}}(\mathcal{G}_{\mathcal{L}(k)}) = \left[ \mathcal{G}_{\mathcal{L}(k)} \right] \text{based is abelian.} \]
\[ \pi_0^{\text{f.}} = \pi_0^{\text{G}_{\mathcal{L}(k)}} = \mathbb{K}^x \]

$\mathcal{G}_{\mathcal{L}(k)}$ is $\mathbb{A}^1$-connected!

\[ (\text{If } n \geq 3) \]
If you can connect any two points by a sequence of copies of $A^r$, (bringing a matrix to $A^r$ for algebraic)

\[
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\]

\[\text{take}\]

0. Why $K(X \times A^1) \cong K(CX)$ for $X$ smooth?

\[\uparrow\]

regular (the regular local rings are regular)

local (the local rings are regular)

Worthington

A local ring $R$ with maximal ideal

is regular local if
\[ \dim R = \dim_{R/m} R/m^2 \]

Always \( \leq \)

Atiyah - Macdonald

**Lemma:** A finitely generated \( R \)-module always has a \textit{finite length resolution} by projective \( R \)-modules if and only if \( R \) is regular. (If it is, the resolution can always be chosen to be of length \( \leq \dim R \).)

\[ Eilenberg-Waxell: \] \( P \) projective \( R \)-module

\[ \exists \text{ free (}\infty\text{-dim.)} \] \( R \)-module \( F \) such that
$p \otimes f \text{ free }$

$p \otimes Q \text{ free }$  $p \otimes Q \otimes Q \otimes p \text{ free }$

With no rank, there is no meaningful $K$-theory.

Left adjoint to forget: $R$-group $\rightarrow$ comm. module

$G(R) = \{ \text{f.g. } R\text{-modules}, \oplus \}$

$G(X) = \{ \text{coherent sheaves}, \oplus \}$

For Noetherian schemes, it turns out $G$ is homotopy invariant.
For every regular scheme $X$, $K(X) \subseteq \mathcal{O}(X)$. 