Theorem: Let $R$ be a Noetherian ring. Then the inclusion $R \to R[t]$ induces a homotopy equivalence

$$G(R) \sim G(R[t]).$$

the infinite loop space associated with the symmetric (generalized) homotopy monoidal category of theory of $p$-spaces) f.g. $R$-modules.

(Recall: For $R$ regular $G(R) \sim K(R)$.)
a f.g. module has a finite length
resolution by f.g. projective modules.

A graded ring (for the moment) means an ungraded
by \( \mathbb{N}_0 \): \( R_m \quad m \in \mathbb{N}_0 \)

\[ R_m \otimes R_k \rightarrow R_{m+k} \quad \text{(commutative, associative, unital)} \quad t \in R_0. \]

\[ G_{\mathbb{N}_0}(R) = \mathbb{K} \{ \text{f.g. graded } R \text{-modules } \}
\]

\[ \bigcup_{k \in \mathbb{N}_0} \quad R_m \otimes R_k \rightarrow R_{m+k} \quad \text{(associative, unital)} \quad k \in \mathbb{N}_0. \]
Lemma: Let $S = A \otimes S_0$, be a graded Noetherian commutative ring which is flat as an $A$-module ($A = S_0$). If $A$ admits a finite $S$-flat resolution, then $A \otimes S$ induces an $A$-equivariant equivalence

$$G(A) \otimes N \xrightarrow{\sim} G^q(S).$$

In particular, there are $\mathbb{Z}[S]$-module isomorphisms

$$G_\ast(A) \otimes \mathbb{Z}[S] \cong G_\ast(S).$$

Proof of the Theorem: Recall we are looking at the inclusion

$$R \rightarrow R[t]$$

(not graded, alternatively, all in degree 0).
Introduce a new variable $\xi$ with deg $\xi = 1$. Consider $R[\xi t, \xi] \subset R[\xi t] \ (\text{graded rings})$

\[ M \rightarrow M / (\xi) M \]

$f$-graded $R[\xi t]$-modules $\sim$ $f$-graded $R[t]$-modules

$\xi$-for some $R[\xi t]$-modules

has the same $K$-theory as its subcategory of $f$-graded $R[t]$-modules on which $\xi$ acts by $0$.

\[ G^g_r(R[\xi t]) \rightarrow G^g_r(R[\xi t, \xi]) \rightarrow G(R[t]) \]
By the lemma, this gives an exact sequence

\[ \mathcal{L}_n(R) \otimes \mathbb{Z}[\delta] \to \mathcal{L}_n(R) \otimes \mathbb{Z}[\epsilon] \to \mathcal{L}_n(R(t)). \]

for each \( n \). This becomes a short exact sequence. \( \square \)

(Details: Quillen's paper on Algebraic K-theory.)

In a similar style, or both periodicity in algebraic K-theory (I mean, the \((2,1)\)-periodicity in K-theory)

(In Voevodsky's language,

\[ \mathcal{L}^{[1]}(\mathcal{L}_{\infty}(k) \times \mathbb{Z}) \cong \mathcal{L}_{\infty}(k) \times \mathbb{Z} \] )
In terms of algebraic K-theory,

\[ K^i(X \times G_m) \cong K^i(X) \oplus K^{i+1}(X) \]

\[ \sum_{i=0}^{\infty} \mathbb{Q}(i) \cong (X \times \mathbb{C}_{\infty})^+ \cong (X \times \mathbb{C}_{\infty}) \vee X^+ \]

\[ \mathbb{N} \to \dim_{\mathbb{Q}}(X) \mathbb{N} \to \dim_{\mathbb{Q}}(X) \mathbb{N} \to \dim_{\mathbb{Q}}(X) \mathbb{N} \]

\[ K^i(X \times \mathbb{P}^1) \cong K^i(X) \oplus K^i(X) \]

\[ \Sigma \mathbb{P}^1 \cong (X_+ \vee \mathbb{P}^1)^+ \vee X_+ \]

Suspension spectrum in motivic category, \[ + \to \Sigma \mathbb{P}^1 \] derived base point.

Actually, after applying one suspension,
\[ \Sigma^0(X \times Y) \sim \Sigma^0(X \cup Y \cup Y) \]

\[ X \cup Y \rightarrow X \times Y \rightarrow X \times X \] (correction)

After suspending:

\[ \varepsilon(X \cup Y) \rightarrow \varepsilon(X \times Y) \]

[\text{Similarly, a retraction is a wedge summand.}]

\[ \forall t \in [0, \frac{1}{2}] \quad (2t, x) \quad \varepsilon(t, x, y) \quad t \in [0, 1] \]

\[ \forall t \in [\frac{1}{2}, 1] \quad (2t-1, y) \]

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Proof of \( \mathcal{O} \): "localization and dévissage" (sheaf)

Let me just consider the case \( X = \text{Spec}(R) \).

Consider an element \( f \in R \). We get a fibration
\[ \text{K}(T_f) \rightarrow \text{K}(R) \rightarrow \text{K}(f^{-1}R) \quad \text{? "localisation exact sequence?} \]

\( T_f \): Category of morphisms \( f : R_1 \rightarrow R_2 \) of finitely generated projective \( R \)-modules, which become an isomorphism after inverting \( f \).

\text{Diagram:} \quad \text{K}(T_f) \sim \text{K}(R/(f)) \quad \text{R}^n \sim \text{R}^n \quad \text{Apply this to} \quad \text{R}[t], \quad f = t

\[ \text{K}(R) \rightarrow \text{K}(\text{R}[t]) \rightarrow \text{K}(\text{R}[t^{-1}]) \rightarrow 0 \]
As generalised chronology the one

$$K(R_{\text{fob}}^{-1}) \sim K(R_{\text{b}}) \sim K(R).$$

(Let’s return to a little more detail next time.)