From last time:

\[ K(T^f) \xrightarrow{\sim} K(A) \rightarrow K(f^*A) \xleftarrow{\text{filtration of loop space}} \]

denotation \( K((A/f^{1})) \)

category of morphisms \( f : P_1 \rightarrow P_2 \)

\( P_1, P_2 \) are f.g. \( A \)-modules, \( y \) after inverting \( f \)

\( A = R[t], \quad f = t \):

\[ K(R) \rightarrow K(R[t]) \rightarrow K(R[t, t^{-1}]) \xrightarrow{(*)} \]

\( \text{Or} \)
\( R : \mathbb{K}[t] \to \mathbb{K}[t'] \to (\mathbb{K}[t'] \to \mathbb{K}[t]) \)

(in the \( \mathbb{K}(R) \to \mathbb{K}(\mathbb{K}[t]) \) different)

\( M \to \mathbb{K}[t] \to \) maps

\( \mathbb{K} \to R[t] \) off inverse

Why \( r \) is 0: Because \( \mathbb{K} \) has a "function:

\[ R[t] \to R[t'/t] \to \mathbb{K} \]

right inverse

\[ \vdash \text{in } \mathbb{K} \text{- theory} \]

\[ \vdash \text{also induces } \]

\[ \vdash \text{induces } \]
mixed

unmixed

(\text{R}_\mathbb{Z} \text{ does not share})

X \leftrightarrow \mathbb{C}, \forall C \in C.

A \leftrightarrow \mathbb{C}, \forall C.

K(\mathbb{Z}, \mathbb{Z}) \sim K(\mathbb{R}) \land \mathbb{Z} \subseteq \mathbb{R}.

(\text{implies: cluster method is a weaker version})
Some more comment on motivic and classical stable homotopy theory:

Classically, $K^*(k) = \mathbb{Z}$ even dimension
0 in odd dimension.

Unplanned set

For any space $X$, there is a construction of $n$-connected cover $X^n$ ($n=1$: the universal cover).
Therefore, we may assume $X$ is simply connected.

Suppose we have constructed $X^m$. By Hurewicz' theorem,

$$
\pi_i X^m = 0 \quad \text{for} \quad i \leq m
$$

There is a map $\phi : X^m \to X$

which induces $\in \pi_i$ for $i > m$.

Suppose $X^{m-1}$ has been constructed. Then by Hurewicz' theorem,

$$
H_i(X^{m-1}; \mathbb{Z}) = 0 \quad \text{for} \quad i \leq m-1,
$$

$$
G = H_m(X^{m-1}; \mathbb{Z}) \cong \pi_m(X^{m-1})
$$

Hurewicz' map.
By the universal coefficient theorem,

\[ H^n(X^{n-1}, \mathbb{Z}) \cong \text{Hom}(\mathbb{Z}, B) \]

This is called the characteristic map represented by \( \phi : X^{n-1} \to K(B, m) \).

The Hurewicz map gives that \( \phi \) induces the identity in \( \pi_m \) (i.e., \( \pi_m(X^{n-1}) = \pi_m(K(B, m)) = 0 \)). Take \( X^n \) to be the homotopy fiber of \( \phi : X^{n-1} \to K(B, m) \).

By the LCS in \( \pi_n \), the composition

\[ X^n \to X^{n-1} \to X \]
induces \( \tilde{i} \) in \( \Pi_k \), \( i > n \).

This is called the Postnikov tower of the space \( X \).

[Postnikov theorem says that if \( X \) is simply connected, then the fiber bundle \( \mathcal{F}(n, m - 1) \rightarrow X^m \rightarrow X^{m-1} \)

\( \cong \)

\( \pi_k \mathcal{F}(n, m) \) topological abelian group

is principal \( G \) classified by a map \( X^{m-1} \rightarrow B\mathcal{F}(n, m-1) \)

\( \cong \pi_k \mathcal{F}(n, m) \).

\( c \in H^m(X^{m-1}; G) \)

This is called the Postnikov invariant. ]
There is a stable analogue:

\[ E^{\text{p spectrum represented by}} \]

\[ \tilde{Z}_n \cong S\tilde{Z}_{n+1}, \quad n \in \mathbb{N} \]

\( n \)

infinite loop spaces

\( k \)-connected cover of \( E \): take the

\((k+1)-\text{connected}\) \( \tilde{Z}_n = (m+k)\)-connected cover

of \( \tilde{Z}_m \)

\[ \tilde{Z}_n \cong S\tilde{Z}_{n+1}, \quad n \in \mathbb{N} \] for a spectrum \( E \).
\[ \pi_i E = 0 \text{ for } i > k \]

\[ 2^n \rightarrow \mathbb{P}^{2n+1} \]

\[ \phi : E^k \rightarrow E \]

The Postnikov tower of spectra.

\[ \text{This yields the Atiyah-Hirzebruch spectral sequence:} \]

\[ E_2^{pq} = H^p(X; E^q(x)) \Rightarrow E^{p+q+2}(X) \]

The homotopy tower of spectra.

Alternate construction: via CW decomposition of \( X \) "conditionally"
Example:

\[ K^*(\mathbb{C}P^n) \]

\[ H^b(\mathbb{C}P^n, \mathcal{O}(1)) = H^{b+2}(\mathbb{C}P^n) \]

\[ d: E_r \rightarrow E_{r+1} \]

\[ E_r^{p,q} = H^{p+q}(X, \mathbb{Z}(r+1)) \]

\[ K^*(\mathbb{C}P^n) = \mathbb{Z}^{\text{even}} \]

\[ \text{for differential} \]
There is an analogue for algebraic $K$-theory.

Instead of the particular tower, we use something called

the slice tower (Block: Higher Chow groups)

the "hard moving lemma" is (possibly)

formal construction

still not known

Fixed by Levine

Which gives a spectral sequence:

To get the indexing, the associated graded pieces
It gives a spectral sequence from Chow groups to algebraic $K$-theory (maybe more detail next time).

This gives an application of Voevodsky's theorem:

Say, you are willing to complete at $k$, say, for a field (or for all varieties)

you can compute Chow groups $CH^d$ completely from Voevodsky's

And then we have a spectral sequence to $K$-theory.
E.g. $K(\mathbb{Q})^3$ is known by this method. localized at $2 : \otimes \mathbb{Z}_{(2)}$. $K \mathbb{Z}$ is finitely generated: $K \mathbb{Z}_2$. (Weibel.)

(maybe even globally)