Constructing Rost varieties

The main step in proving $H^0(m)$ by induction on $m$:

Theorem: Suppose $H^0(m-1)$ holds. Then for every field $k$
of characteristic 0 and every non-zero symbol

$$\alpha = (\alpha_1, \ldots, \alpha_m) \in \text{End}_k(k)/\ell$$

there exist a smooth projective variety $X_\alpha$ whose function field

$k(X_\alpha)$ satisfies.
(a) \[ a = 0 \in K^m_\mu (k_a)/l \]

(b) The map \[ H^{m'}(k, \mathcal{Z}(v)(m)) \rightarrow H^{m' + n}_{\text{et}}(k_a, \mathcal{Z}(v)(m)) \]

is injective.

The main step in constructing the varieties \( X_n \).

(Generally, (a) comes as a constructed property of \( X_n \).)

The difficulty is in proving (b). Here the main point is applying cohomology operations to get to a cohomology group in a more convenient dimension.

So the variety \( X_a \) must combine some arithmetic properties with some topological properties.
The topological properties: (a story related to complex cohomology and algebraic cohomology)

Discussion (warm-up)

Complex cohomology

Compact smooth manifolds $M$ of a given dimension $n$

Stably weakly complex structure: complex $\mathbf{N}$-bundle $\mathcal{E}$

an isomorphism $\mathcal{E} \otimes \mathcal{T}_M \cong \mathbf{IR}^N$

$a$ part of the structure
\( M \sim 0 \) if \( M \) is a boundary of a \((n+1)\)-dim.

weakly almost complex manifold \( \mathcal{P} \) in a

way which respects the \( \mathcal{P} \) is a weakly

complex structure.

\[ M_1 \sim M_2 \]

\[ M_1 \cup M_2 = \mathcal{P} \]

same as above

\[ \text{in-bound} \]

\[ \text{out-bound} \]

\[ \text{collar vector on } M_2 \]

\[ \text{collar vector on } M_1 \]
$m$-dim almost complex manifolds $\mathcal{M}$

$\omega^m_n$ complex extension

\[ \Omega^\mathcal{C}_n = \bigoplus_{\alpha > 0} \mathbb{C}_n \times \alpha \text{ is a cone under } \mathbb{C}_n \times \mathbb{R}. \]

Note: There is a spectrum (topological) $MU$

Pre-spectrum:

$D_{2m} := BU(n) \uparrow \text{ Thom space}$

Thom space of a bundle
\[ \sum \mathbb{B}U(n)^{r_n} \rightarrow \mathbb{B}U(m+1) \] on \( X \). If \( X \) compact, 1-point compactification of total space of the bundle.

\[ S^{k+1} \mathbb{B}U(n) = \mathbb{T}_n \oplus 1 \]

For \( X \) general:

\[ \lim_{\mathbb{K} \xrightarrow{k} 0} \mathbb{K} \subseteq X \]

This spectrum is called \( \overline{MU} \).

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In \( \mathbb{A}^1 \)-homotopy context, we have an analogous spectrum \( \Omega \mathbb{B}U \):

\[ \Omega \mathbb{B}U_0 \rightarrow \mathbb{B}U(k) \)

from argument goes through algebraically.
The associated spectrum is called $MU$.

Back to topology:

Pontrjagin (Theorem): $\Pi_n MU \cong \Omega^n$  

Stably almost all  

identification of $S^{N+n}$. 

$S^N \rightarrow M \overset{\xi}{\rightarrow} \Omega^n MU$
Going back: A transversality (Milnor)

It fails in general in enriched context
- $\sigma$-invariant true in
- algebraic dimension

Levine (Levine - Pandharipande)
Calculating $\pi_* MU = \Omega^C$ is a separate issue.

Before calculating homotopy, let's calculate integral homology. Maybe, let's try to study homotopy by the Hurewicz homomorphism, $\pi_* \to H_*(\mathbb{C}, \mathbb{Z})$.

$$H_\ast(MU, \mathbb{Z}) \cong H_\ast(BU, \mathbb{Z})$$

In topology,

- $U_m \cong GL_m \mathbb{C}$
- $U \cong GL \mathbb{C}$

where $|c_i| = 2i$.
An element of the cohomology of $B\mathcal{G}$ gives an invariant of principal $\mathcal{G}$-bundles:

\[ \alpha \in H^\ast X \leftrightarrow H^\ast B\mathcal{G} \]

\[ \alpha(\xi) \in \mathcal{X} \rightarrow B\mathcal{G} \]

$\xi$-bundle $\xi$ on $X$ classifying map

The fibre-wise homomorphism $\mathcal{L} \in \alpha$ polynomial on chain classes $\alpha \in H^\ast NU = H^\ast BU$

\[ \mathcal{L}^\alpha : \prod \mathcal{I} \rightarrow H^\ast NU \rightarrow \mathbb{Z} \]
$\mathbb{U}$

$\mathbb{M} \rightarrow \alpha [M]$

\[ \langle \alpha (\mathcal{F}), [M] \rangle \]

\[ \nu = \text{stable normal bundle of } \mathbb{N} \]

Theorem (Milnor - Novikov): The Hurewicz homomorphism

\[ S^1 \simeq \pi_1 \mathbb{M} \xrightarrow{h} \text{Het} \mathbb{M} = \mathbb{Z} \left[ b_1, b_2, b_3, \ldots \right] \]

\[ |b_i| = 2 \nu \]
is injective.

$H_k \mathbb{C}P^n \cong [-2]$

Milnor-Stasheff:
Characteristic classes

It is onto in all dimensions except $2l^2 - 1$, $l$ prime. In those dimensions, the image is generated by

\[ h \text{ mod decomposables.} \]

A manifold which has this generator \( (\mathbb{Z}/2 \text{ - information}) \)\) is called a Milnor manifold. (Milnor characteristic number)
The topological property of most varieties is that they are algebraic analogues of Milnor manifolds.