Let $X$ be a Rost variety.

Motives (Murre, Voevodsky, Weibel)

$\mathbb{H} \otimes \mathbb{Z}$: integral motive homology spectrum

Motivic symmetric spectra: same derived category as motivic spectra, but smash product is a symmetric monoidal
Picks up a model structure: equivalence \( \mathcal{N} \sim \mathcal{M} \) of symmetric spectra

\( \text{Cofo connected \Rightarrow weak equivalence} \)

\( \text{cell = the model on a spectrum alg.} \)

\( R^2 \)

(Note: In topology, you could define chain complexes..."
as $H^1$-modules. In the analogous model structure, the derived category is the derived category of $\mathcal{D}$. 

\[ \mathcal{D} H^1 \text{-modules} \rightsquigarrow \mathcal{D} R. \left\langle \text{Written down by Handel (possibly his first here?)} \right. \]

In the $A^1$-context, (Voevodsky) motives are substitutes for chain complexes.

Let $X$ be a $R$-test variety. Recall the Čech resolution of $X$: simplified object $\mathcal{C}(X)$.
With simplicial stage: \( X^{m+1} \)
given by projections

\[ \text{faces: } X \rightarrow \ast \]

Degeneracies: given by diagonal maps

\[ \Delta : X \rightarrow X \times X \]

(Apply in one coordinate each)

(e.g. if \( X \) is a group, \( \tilde{C}(G) \subset B(C/G, \ast) \))

\[ \tilde{C}(G) \cong B(C/G, \ast) \]

(dg) homogeneity:

\[ (g_0, g_0, g_1, \ldots, g_0, \ldots, g_n) \leftarrow (g_0, \ldots, g_n) \]

If \( X \) has a point (such as is the case of a group)
then $\mathcal{C}(X) \sim *$.

But if $X$ has no point (such as in the case of $\text{Spec } k$ a field or variety) then $\mathcal{C}(X)$ has no point, so $\not\exists* \text{ in the } A^1$-homotopy category. Nevertheless, we think of $\mathcal{C}(X)$ as being "point-like".

Convention: $d = \ell^{m-1}$

$$b = \frac{d}{\ell - 1} = 1 + \ell + \cdots + \ell^{m-2}.$$ 

The motive of an $A^1$-space $X$ is $\mathbb{H}Z_{\text{mot}}(X)$. In the derived category.
Theorem: The motive of a Rost variety contains a $V$-motive (called the Rost motive) $M$ which satisfies the following: (H. Dercks, W. Rost)

\begin{align*}
(1) & \quad M^* (d) \sim M, \quad (\text{Grothendieck-Chow motives}) \\
& \uparrow \\
& F_{H \otimes} (\mathbb{N}, H \otimes) \vee F (d), \\
& \downarrow \text{purity} \\
& H_2 \otimes \vee \mathcal{D}_m \\
(2) & \quad \text{There exist two cofibration sequences in the category of Voevodsky motives}.
\end{align*}
\[ D(b) \rightarrow M \rightarrow H \gamma_{\nu_{\mu}} \wedge \check{\mathcal{C}}(X) \]

\[ H \gamma_{\nu_{\mu}} \wedge \check{\mathcal{C}}(X)(d) \rightarrow M \rightarrow D. \]

**Lemma:** Assume \( H_{90}(n-1) \). Then

\[ H^{m+1,\nu}(\check{\mathcal{C}}(X), \mathcal{Z}_{(s)}) = 0. \]

**Proof sketch:** Use the Milnor operations \( Q_1, \ldots, Q_{m+1} \). One proves that
\[ Q_{n-1} Q_{n-2} \cdots Q_1 : \tilde{H}^{n+1, m}(\tilde{C}(X), \mathbb{Z}_e) \to \tilde{H}^{2n+2, 2n+1}(\tilde{C}(X), \mathbb{Z}_e) \]

\[ a_i : 2l - 1, \ell - 1 \]

\[ \sum_{i=1}^{m-1} (2l_i - 1, \ell_i - 1) = \left( 2 \frac{\ell^{m-1}}{\ell-1} \ell - m + 1, \frac{\ell^{m-1}}{\ell-1} \ell - m + 1 \right) \]

The main induction step: Suppose \( X \) is a Rost variety and \( H^{0,0}(n-1) \) holds. Then

\[ H^{n+1, m+1}(k, \mathbb{Z}_e(m)) \to H^{n+1, m+1}(k(X), \mathbb{Z}_e(m)) \]
ii) an injection.

**Notation:** $\pi$ the geometric morphism of toposes from the Nisnevich to the étale topos.

(From (i))

$$T(m) := \tau \leq m+1 \Rightarrow \pi_* \left( \mathcal{Z}(\mu) \otimes \mu \right).$$

We have a cofiber sequence of motives:

$$\mathcal{Z}(\mu) \rightarrow T(m) \rightarrow K(m) \leq \text{(this is not)}$$

**Remark:** $K$-theory.
We have a commutative diagram of groups:

\[ 0 = H^{n+1}_{\text{not}}(\tilde{\mathcal{C}}(X), \mathbb{Z}(e)(n)) \rightarrow \bigwedge_{\text{not}} H^{n+1}_{\text{not}}(\tilde{\mathcal{C}}(X), \mathbb{Z}(e)(n)) \rightarrow H^{n+1}_{\text{not}}(\tilde{\mathcal{C}}(X), \mathbb{Z}(e)(n)) \rightarrow \bigwedge_{\text{not}} H^{n+1}_{\text{not}}(\tilde{\mathcal{C}}(X), \mathbb{Z}(e)(n)) \rightarrow 0 \]

\[ H^{n+1}_{\text{not}}(\tilde{\mathcal{E}}, T(e)(n)) \cong H^{n+1}_{\text{not}}(\tilde{\mathcal{E}}, \mathbb{Z}(e)(n)) \quad \tilde{\mathcal{E}} = \text{Spec } k(X) \]
\[ \tilde{C}(X) \rightarrow \text{Spec } k \]

...over a large enough algebraic extension, ~

...but in et, we have Galois descent.

It suffices to show that the right vertical arrow is an injection.
\[ E = \delta_{m+1} \cdot \mathcal{K}(X) \]

\[ \mathcal{Z}(e) \rightarrow \sum_{m+1} R_{\pi+}(\mathcal{Z}(e) \cdot \mathcal{K}(m)) \]

I will finish this next time.

(I would also like to show the construction of norm varieties.)