Note Title 8/15/2013

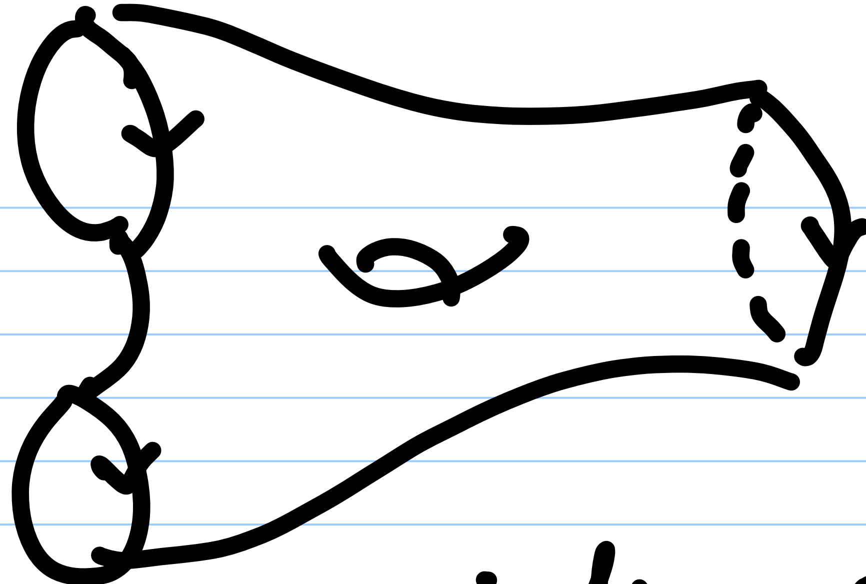
MODULAR FUNCTORS

COINED BY G. SEGAL

RIEMANN SURFACE :

ANALYTICALLY PARAMETRIZED

∂ COMPONENTS



i · dp

outside
<u>outbound</u>
inside
<u>inbound</u>

MODULAR FUNCTOR.

FINITE SET S OF LABELS

S-labelled Riemann surface Σ

$\longmapsto$

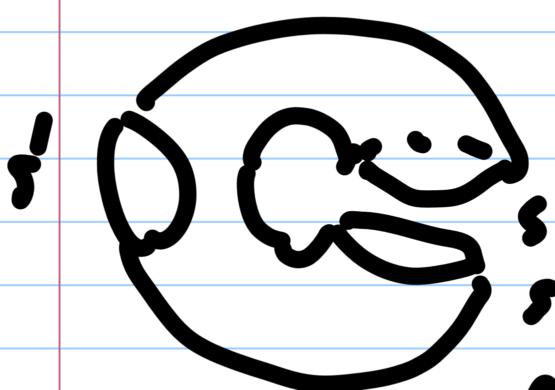
f.d. $\mathbb{C}$-vector space M_Σ
(depend holomorphically on Σ)

- $M_{\Sigma \sqcup T} \cong M_\Sigma \otimes M_T$

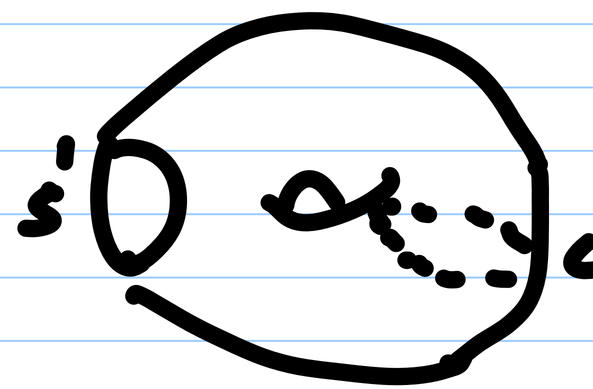
•

$\Sigma_s^c \leftarrow \Sigma$ cut along c ← both new comfonent labeled s

c Jordan curve in Σ^0



Σ_s^c



Σ

$$M_\Sigma \cong \bigoplus_{s \in S} M_{\Sigma_s^c}$$

Example of a coherence relation:

$$(*) \quad \begin{tikzcd}
M_{\Sigma \amalg T} \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\wr\!\wr"] & M_\Sigma \otimes M_T \arrow[d, phantom, "\wr\!\wr\ \text{switch}"] \\
M_{T \amalg \Sigma} \arrow[r, phantom, "\cong"] & M_T \otimes M_\Sigma
\end{tikzcd}$$

(non-)
Example: Quillen determinant

if Σ_1 has no closed components:

$$\begin{tikzcd} \mathrm{Hol}(\Sigma) \arrow[r, "\subseteq"] & \Omega^0(\partial\Sigma) \arrow[d, "\mathrm{proj}^{\perp}"] \\ & \Omega^0_+(\partial\Sigma) \end{tikzcd}$$

α. holomorphic functions

$$\Omega^0(S^1) \cong \Omega^0_+(S^1) \oplus \Omega^0_-(S^1)$$

$$\begin{array}{cc} \| & \| \\ \langle 1, z, z^2, \dots \rangle & \langle z^{-1}, z^{-2}, \dots \rangle \end{array}$$

$$F_\Sigma : \mathrm{Hol}(\Sigma) \longrightarrow \Omega^0_+(\partial\Sigma)$$

Fredholm

$$\mathrm{Det}\,F = \mathrm{Det}(\mathrm{Ker}\,F)^{-1} \otimes \mathrm{Det}(\mathrm{Coker}\,F)$$

(Pressley - Segal : Loop groups)

$\mathrm{Det}\,\Sigma := \mathrm{Det}\,F_\Sigma.$

Σ closed $\Rightarrow$ $\mathrm{Det}\,\Sigma :=$

$$\big(\mathrm{Det}\,(H^0(\Sigma, \underline{\mathrm{Hol}}))\big) \otimes \mathrm{Det}\,(H^1(\Sigma, \underline{\mathrm{Hol}}))^{-1}$$

$\uparrow$ top Ext. power

$S = \{1\}$

- Normalization condition

D

$$M_{D_1} = \mathbb{C}$$

$$M_{D_s} = 0$$

$s \neq 1.$

DELIGNE (~ 1999)

NOTICED THAT CONDITION (*)

CAUSES PROBLEMS:

DET (V) is A $\underbrace{\text{SUPER-LINE}}_{\text{LINE}}$

f.d.

even or odd

If L, M ODD,

$$L \otimes M \xrightarrow{\text{switch}} M \otimes L.$$

$-$

2003 (K. On Spin and modularity ...)

DELIGNE FOUND AN EVEN WORSE EXAMPLE

"CHIRAL FERMION"

RIEMANN SURFACES WITH SPIN STRUCTURE.

$$L \otimes L \cong T\Sigma$$
$$\Omega^{1/2}\Sigma$$

ON S^1, THERE ARE TWO SPIN STRUCTURES:

MÖBIUS STRIP ANTI-PERIODIC

PERIODIC (TRIVIAL)

 A

ON $\Omega^{1/2}S^1$, there is an antisymmetric pairing

$$S(\eta, \xi) = \int \eta \xi$$

$$\Omega^{1/2}_{Hol}(\Sigma) \subset \Omega^{1/2}_{\partial\Sigma}$$

ISOTROPIC CS trivial

there)

LAGRANGIAN (MAXIMAL ISOTROPIC)

$$F_{1/2} : \begin{tikzcd} \Omega^{1/2}_{Hol}(\Sigma) \arrow[r] \arrow[rr, bend right, "\text{FREDHOLM}"'] & \Omega^{1/2}_{\partial\Sigma} \arrow[r] & \Omega^{1/2}_{+\partial\Sigma} \end{tikzcd}$$

$\Rightarrow$

$$\mathrm{Pf}\, F_{1/2} =: \Phi_{\Sigma}$$

WHEN WE LOOK AT THE PERIODIC SPIN STRUCTURE ON S^1,

$$S(\eta, \xi) = \int \eta \xi$$

has null space of dim 1.

"THROW THIS OUT"

$$C(x_1, \cdots x_n) = T_{\mathbb{C}}(x_1, \cdots x_n) \Big/ \begin{matrix} x_i^2 = 1 \\ x_i \cdot x_j = -x_j \cdot x_i \\ i \neq j \end{matrix}$$

Φ_e = CLIFFORD MODULE
(MORITA EQ. WITH $\mathbb{C}$)

OVER C(number of periodic

∂ components)

even

Deligne : Clifford modules,
1-dim. Morita equivalences
modular functors
(just one label)

K., LAI: GENERALIZE DEFINITION TO A SET OF LABELS

⟿ REALIZATION INTO (TWISTED) K-MODULES

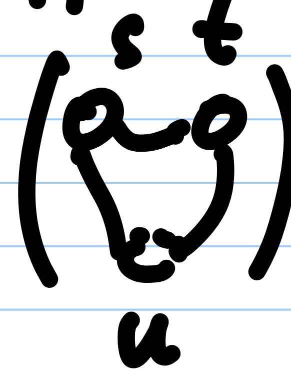
VERLINDE ALGEBRAS

(RINGS)

$\mathbb{Z} S$

set of labels

$$s \cdot t = \sum_u \left(\dim M \right) u \in \mathbb{Z}$$

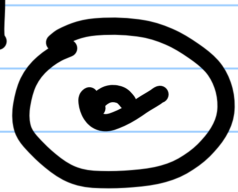


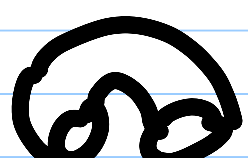
MATH. PHYSICISTS: VERLINDE RING IS A GLOBALLY

COMPLETE INTERSECTION RING.

VERLINDE CONJECTURE :

NOTE :

M  $= \mathbb{C}S.$



$-1/\tau$: diagonalizes the product matrix.

PROOF CLAIMS:

MOORE-SEIBERG

... HUANG.

EXAMPLE (FIORG, K.):

MODULAR FUNCTORS

FROM EVEN LATTICES

$L \subset \mathbb{R}^n$

2) $\langle x, x \rangle$, $x \in L$

VERLINDE RING:

$\mathbb{Z}(L^* / L)$.

EXAMPLE (STILL MILDLY

CONJECTURAL)

G = SIMPLY CONNECTED
SEMISIMPLE COMPACT LIE
GROUP (CHIRAL WZW
MODEL)

$$1 \to S^1 \to \widetilde{LG} \longrightarrow LG \to 1$$

$\nearrow$ 2-CONNECTED

$LG = \mathrm{Map}(S^1, G)$

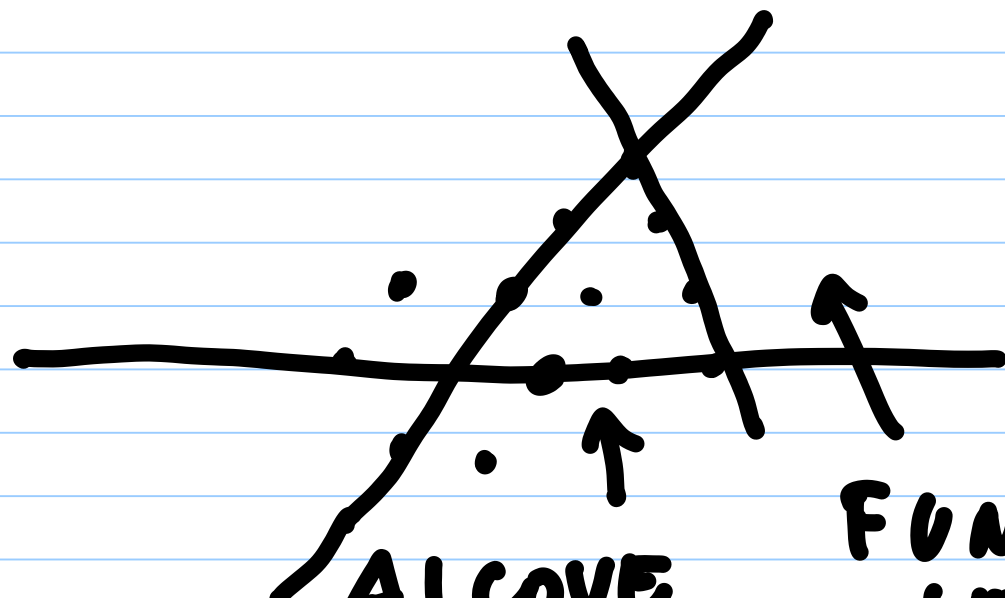
(PRESSLEY - SEGAL : LOOP GROUPS)

A FINITE NUMBER OF LOWEST WEIGHT IRREDUCIBLE REPS, HILBERT OF $\widetilde{LG}$

WHERE CENTER ACTS BY A GIVEN WEIGHT (LEVEL k)

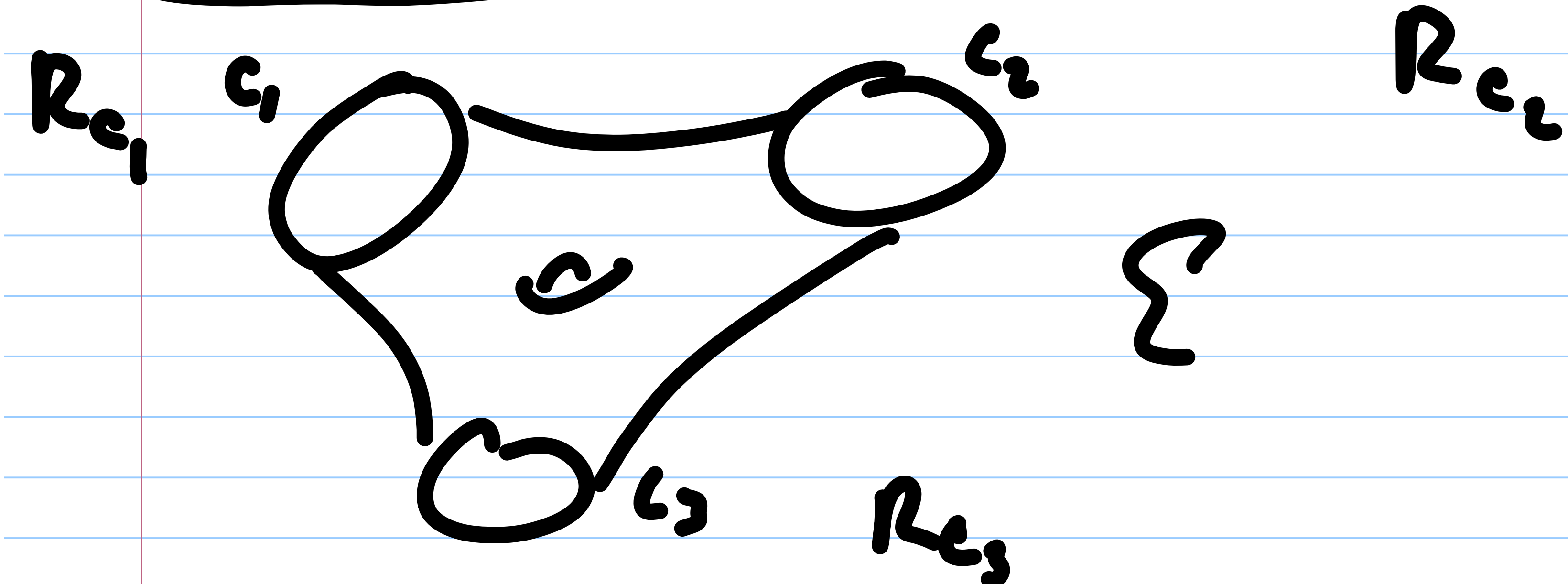
THESE REPS. = LABELS OF A MODULAR FUNCTOR

AFFINE ALGEBRA REPS.



$\langle \ , \text{highest root} \rangle \leq k$

REPS. (LABELS) = WEIGHTS INSIDE THE ALCOVE.



$$R_{c_1} \hat{\otimes} \cdots \hat{\otimes} R_{c_m} = \text{rep. of } \widetilde{\mathrm{Map}(\partial\Sigma, G_{\mathbb{C}})}$$

CENTRAL EXTENSION WHOSE COCYCLE SPLITS CANONICALLY ON

$$\mathrm{Hol}(\Sigma, G_{\mathbb{C}}).$$

?

(pass to Lie algebras, Stokes theorem)

$$M_\Sigma := \left(\bigotimes_{c \,\partial\, \text{comp.}} R_c \right)^{\mathrm{Hol}(\Sigma, G_\theta)}.$$

(BAKALOV-KIRILLOV, HUANG PROOF CLAIMS)

? FOR Σ CLOSED

$\Rightarrow$

NON-ABELIAN θ-BUNDLES

(EASY COHOMOLOGICAL

DESCRIPTION?)

VERLINDE(S) USED VERLINDE

CONJECTURE TO DERIVE THE

(FOR W2W MODEL)

<u>VERLINDE FORMULA</u>

COUNTING NON-ABELIAN

θ-FUNCTION.

FREED - HOPKINS - TELEMAN

$$V \cong K_{G,k+h^\vee}(G)$$

EQUIVARIANT TWISTED K-THEORY