THE HOMOTOPY LIMIT PROBLEM FOR HERMITIAN K-THEORY, EQUIVARIANT MOTIVIC HOMOTOPY THEORY AND MOTIVIC REAL COBORDISM

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1. Introduction

The homotopy limit problem for Karoubi's Hermitian K-theory [23] was posed by Thomason in 1983 [43]. There is a canonical map from algebraic Hermitian K-theory to the $\mathbb{Z}/2$-homotopy fixed points of algebraic K-theory. The problem asks, roughly, how close this map is to being an isomorphism, specifically after completion at 2. In this paper, we solve this problem completely for fields of characteristic 0 (Theorems 16, 20). We show that the 2-completed map is an isomorphism for fields F of characteristic 0 which satisfy $cd_2(F[i]) < \infty$, but not in general.

The main ingredient of our method is developing G-equivariant motivic stable homotopy theory for a finite group G. Our particular emphasis is on $G = \mathbb{Z}/2$, and on developing motivic analogues of Real-oriented homotopy theory along the lines of [17]. Karoubi's Hermitian K-theory can be shown to be a $\mathbb{Z}/2$-equivariant motivic spectrum in our sense. This can be viewed as an algebraic analogue of Atiyah's Real $K\mathbb{R}$-theory [1]. Viewing Hermitian K-theory in this way is crucial to our approach to the homotopy limit problem, as the solution uses a combination of equivariant and motivic techniques (such as the Tate diagram and the slice spectral sequence).

There are other benefits of equivariant stable motivic homotopy theory, such as constructions of interesting motivic analogues of other Real-oriented spectra, notably a motivic analogue $MGL\mathbb{R}$ of Landweber's Real cobordism E_∞-ring spectrum $M\mathbb{R}$ ([27, 17]). Applying "geometric fixed points" to $MGL\mathbb{R}$ also allows the construction of a motivic analogue of the non-equivariant spectrum MO, which was a question asked by Jack Morava. These constructions however lead to many new open questions, and a thorough investigation of these new motivic spectra will be done in subsequent papers.

To present our results in more detail, we need to start with the foundations of G-equivariant motivic stable homotopy theory, which in turn requires unstable G-equivariant motivic homotopy theory. We work over fields of characteristic 0. In the unstable case, there are foundational notes [44], but our motivation is somewhat different. In [44], a part of the motivation is to be able to take quotient spaces, with the particular example of symmetric products in mind (which, in turn, is needed in studying motivic Eilenberg-MacLane spaces). In the present paper, we do not focus on taking quotients with respect to the group G, but are instead more interested in taking fixed points, which is closer to the context of G-equivariant (stable) homotopy theory of spaces. Because of this, we may stay in the category of (separable) smooth G-equivariant schemes, and we can take more direct analogues of the definitions of Nisnievich topology and closed model structure in the non-equivariant case.

When stabilizing, however, an important question is what is the "sphere-like object" we are stabilizing with respect to, as clearly several potentially natural choices may arise. The answer we give in this paper is to stabilize with respect to the "one-point compactification" $\mathbb{T}_G$ of the regular representation $\mathbb{A}^G$ of the group G. Again, we can then mimic most the construction of the motivic stable homotopy category in the non-equivariant case, as presented, for example, in [20].

In equivariant stable homotopy theory, the basic tools [30] are the Wirthmüller isomorphism (i.e. equivariant stability with respect to finite G-sets), Adams isomorphism and the Tate diagram. We give here motivic analogues of these tools at least in the basic cases. One of the important features of the theory is that the correct motivic analogue of the free contractible G-CW complex EG in this context is again the simplicial model of EG (rather than other models one could potentially think of, such as EG_{et}).

As mentioned above, our first main application is a presentation of Karoubi's Hermitian K-theory as a $\mathbb{Z}/2$-equivariant motivic spectrum $K\mathbb{R}^{alg}$ in our sense. For $G = \mathbb{Z}/2$, we find that $\mathbb{T}_G$ decomposes to a smash product of four different 1-spheres, namely S^1 and $S^\alpha = \mathbb{G}_m$ with trivial action, and S^γ, which is a simplicial model of S^1 with the sign involution, and $S^{\gamma\alpha}$, which is $\mathbb{G}_m^{1/z}$, i.e. $\mathbb{G}_m$ with the involution $z \mapsto 1/z$. Generalizing the methods of Hornbostel [14], we prove that we indeed have a $\mathbb{Z}/2$-equivariant motivic spectrum $K\mathbb{R}^{alg}$ which enjoys three independent periodicities, namely with periods $\alpha + \gamma$, $4 - 4\gamma$, $1 + \gamma\alpha$ (the first two of which are essentially proved in [14]).

Using this machinery, we prove that the inclusion $c : S^0 \to S^\gamma$ is homotopic to $\eta : S^\alpha \to S^0$ in the coefficients of $K\mathbb{R}^{alg}$, which answers a question of Hornbostel [14]. It also gives one form of an answer to the completion problem for Hermitian K-theory: we prove that the Borel cohomology of Hermitian K-theory is its completion at η. However, one may ask if Hermitian K-theory coincides with its Borel cohomology when completed at 2 (there are many partial results in this direction, e.g. [25, 5, 6]). We show that this is false for a general field, but is true for characteristic 0 fields satisfying $cd_2(F[i])) < \infty$. Examples include fields of finite transcendence degree over $\mathbb{Q}$, and $\mathbb{R}$.

The other main focus of the present paper is a $\mathbb{Z}/2$-equivariant motivic spectrum $MGL\mathbb{R}$ which is an analogue of Landweber's Real cobordism $M\mathbb{R}$. The existence of such a spectrum is strongly motivated by Hermitian K-theory. We construct such a spectrum, and further show that it is a $\mathbb{Z}/2$-equivariant motivic E_∞ ring spectrum. There are many interesting implications of this fact. Taking geometric fixed points for example gives a motivic analogue of unoriented cobordism, which answers a question of Jack Morava. Even more interestingly, however, there is a theory of motivic Real orientations, analogous to the theory of [17]. A motivic Real orientation class occurs in dimension $1 + \gamma\alpha$, and is present both for motivic Real cobordism and for Hermitian K-theory. Further, motivic Real orientation gives a formal group law, and hence a map from the Lazard ring to the coefficient ring. In the case of $MGL\mathbb{R}$, one can then use this to apply the constructions of [9] to construct motivic analogues of the "Real spectra series" of [17], including, for example, motivic Real Johnson-Wilson spectra and motivic Real Morava K-theories. It is worthwhile remarking that one therefore has the ability to construct motivic analogues of the various spectra which figure in Hill-Hopkins-Ravenel's recent paper on the non-existence of Kervaire invariant one elements [13], although the exact role of these $\mathbb{Z}/2$-equivariant motivic spectra is not yet clear.

The present paper is organized as follows: Foundations of unstable and stable G-equivariant motivic homotopy theory in our setting are given in Section 2. The Wirthmüller and Adams isomorphisms and the Tate diagram are presented in Section 3. The work on Hermitian K-theory and the completion problem is in Sections 4 and 5. The results on motivic Real cobordism are in Section 6.

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2. The foundations of equivariant stable motivic homotopy theory

2.1. **The site.** Throughout this paper, we shall work over a base field k of characteristic 0. We begin with the foundations of equivariant unstable motivic homotophy theory. Our definitions are different from those of [44]. The main reason is that, similarly as in developing equivariant stable homotopy theory in topology, our emphasis is not on the functor of taking quotient by the action of the group, but rather on taking fixed points. Therefore, we gear our foundations toward making fixed points (rather than quotients) behave well.

In this paper, we will consider the site $S(G)_{Nis}$ of G-equivariant separated smooth schemes over k with the Nisnievich topology, where G is a finite group. In our definition, the covers in the G-equivariant Nisnievich topology are G-equivariant étale maps f in which for each point x (in the étale sense) with isotropy group $H \subseteq G$, there exists a point in $f^{-1}(x)$ with the same residue field and the same isotropy group. Note that for such X, X^G is smooth closed: to show X^G is smooth, consider an affine cover (U_i) of X. Then

$$(1) \qquad \left(\bigcap_{g\in G} gU_i\right)$$

is a cover of X^G by open affine sets in X (because X is separated). In this setting, we have Luna's slice theorem [31], which shows that taking G-fixed points in each of the sets (1) gives a closed smooth subscheme.

By the category of *based G-equivariant motivic spaces* we shall mean the category $\Delta^{Op} - Sh.(S(G)_{Nis})$ of pointed simplicial sheaves on the site $S(G)_{Nis}$.

It may be worthwhile to point out that this category passes a trivial but important test: it captures arbitrary G-sets. In effect, recall that the category of G-sets and G-equivariant maps is equivalent to the category of presheaves (of sets) on the orbit category $Orb(G)$, i.e. the category of transitive G-sets and equivariant maps. For a G-set S, the presheaf on $Orb(G)$ is

$$G/H \mapsto S^H,$$

and for a presheaf F on $Orb(G)$, the corresponding G-set is

$$G/? \times_{Orb(G)} F$$

where $G/?$ is considered as a covariant functor $Orb(G) \to G$-sets.

It worth pointing out however that some constructions which are obvious on G-sets actually require a moment of thought on G-equivariant motivic spaces as defined here. For example, the *forgetful functor* from G-equivariant motivic spaces to H-equivariant motivic spaces, $H \subseteq G$, is obtain by restricting the sheaf to $G \times_H ?$ where the variable ? indicates an H-equivariant separated smooth scheme.

On the other hand, for a normal subgroup H of G, the G/H-equivariant motivic space X^H is modelled simply by restricting X to H-fixed schemes.

On this category, we can put a closed model structure as follows (this is the original, now called "injective", model structure which Joyal described in his 1984 letter to Grothendieck, see also Jardine [21]): The *(simplicial) equivalences* are local equivalences in the sense of [44], i.e. maps of pointed simplicial sheaves $F_* \to F'_*$ which induce an isomorphism on π_0, and for each local section u of F_0, an isomorphism on $\pi_i(?, u)$. Here π_i are the sheaves associated with the presheaves of homotopy groups (sets for $i = 0$) of the simplicial sets obtained by taking sections of the argument over a given object of the site.

The *cofibrations* are simply injective maps on sections; as usual, this specifies fibrations as morphisms satisfying the right lifting property with respect to acyclic cofibrations.

The $\mathbb{A}^1$-model structure is obtained by localizing with respect to projections

$$X \wedge \mathbb{A}^1_+ \to X \quad \text{for } X \in Obj \Delta^{Op} - Sh_\cdot(S(G)_{Nis}).$$

The homotopy categories of the simplicial (resp. $\mathbb{A}^1$-) model structures on $\Delta^{Op} - Sh_\cdot(S(G)_{Nis})$ will be denoted by $h_s(G)$, $h_a(G)$, respectively.

Lemma 1. *Let V be a G-representation and $X \in Sh_\cdot(S(G)_{Nis})$. Then the projection*

$$\pi : X \wedge V_+ \to X \tag{2}$$

is an $\mathbb{A}^1$-equivalence.

Proof: (2) has an "$\mathbb{A}^1$-homotopy inverse", namely the zero-section map

$$q : X \to X \wedge V_+.$$

We have $\pi q = Id_X$, and there exists an "$\mathbb{A}^1$-homotopy"

$$h : \mathbb{A}^1_+ \wedge X \wedge V_+ \to X \wedge V_+$$

where $h(0, u) = u$, $h(1, u) = q\pi$. Under such circumstances, π and q are inverse in the $\mathbb{A}^1$-homotopy category for formal reasons. □

2.2. **Stabilization.** The first question in equivariant stable homotopy theory always is what to stabilize with respect to. In this paper, we stabilize with respect to the "one point compactification of the regular representation". For an affine space V, denote the corresponding projective space by $P(V)$. Then we put

$$S^V := P(V \oplus \mathbb{A}^1)/P(V). \tag{3}$$

Next, put

$$\mathbb{T}_G = S^{\mathbb{A}^G}. \tag{4}$$

The category of equivariant motivic spectra is then defined analogously as in [20]: By $\mathbb{T}_G$*-spectra* (or simply *G-equivariant motivic spectra or G-*$\mathbb{A}^1$*-spectra*) we shall mean sequences (X_n) of based motivic G-spaces together with structure maps

$$\mathbb{T}_G \wedge X_n \to X_{n+1}. \tag{5}$$

Morphisms of spectra are just morphisms in the category of diagrams formed by the objects X_n and morphisms (5).

Similarly as in Jardine [20], to make the construction work, we need the following result:

Lemma 2. *The switch*

$$T_\sigma : \mathbb{T}_G \wedge \mathbb{T}_G \wedge \mathbb{T}_G \to \mathbb{T}_G \wedge \mathbb{T}_G \wedge \mathbb{T}_G \tag{6}$$

induced by the cyclic permutaiton σ *of* 3 *elements is G-equivariantly* $\mathbb{A}^1$*-homotopic to the identity.*

Proof: We shall construct a G-equivariant linear $\mathbb{A}^1$-homotopy between

$$Id, T_\sigma : \mathbb{A}^{3G} \to \mathbb{A}^{3G}. \tag{7}$$

This can be accomplished by taking $Id_{\mathbb{A}^G}$ and tensoring it with a sequence of elementary row operations converting the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now (6) can be identified as a "one point compactification" of T_σ in (7), a notion made precise in the standard way using resolution of singularities. □

The *level-wise model structure* on G-equivariant motivic spectra is defined so that

$$(X_n) \to (Y_n)$$

is a fibration, resp. equivalence if and only if each of the constituent maps

$$X_n \to Y_n$$

is a fibration, resp. equivalence in the $\mathbb{A}^1$-closed model structure on based motivic G-spaces. Cofibrations are defined as maps satisfying the left lifting property with respect to acyclic fibrations.

Letting

$$j_X : X \to JX$$

be natural level-wise fibrant replacement, the *stable model structure* has as equivalences (called *stable equivalences*) maps

$$g : X \to Y$$

where

$$Q_{\mathbb{T}_G} J(g) : Q_{\mathbb{T}_G} JX \to Q_{\mathbb{T}_G} JY$$

is a level-wise equivalence where $Q_{\mathbb{T}_G}$ is stabilization with respect to *shift suspension* $\Sigma'_{\mathbb{T}_G}$.The shift suspension is defined by

$$(\Sigma'_{\mathbb{T}_G} X)_n = \mathbb{T}_G \wedge X_n$$

and the structure maps are

$$\mathbb{T}_G \wedge (\mathbb{T}_G \wedge X_n) \xrightarrow{T} \mathbb{T}_G \wedge (\mathbb{T}_G \wedge X_n) \longrightarrow \mathbb{T}_G \wedge X_{n+1}.$$

where T is the map switching the two $\mathbb{T}_G$ coordinates and the second map is $Id_{\mathbb{T}_G}$ smashed with the structure map of X.

If we denote by $\Omega'_{\mathbb{T}_G}$ the right adjoint to the functor $\Sigma'_{\mathbb{T}_G}$, then the functor $Q_{\mathbb{T}_G}$ is defined as

$$\varinjlim \Omega'^n_{\mathbb{T}_G} \Sigma'^n_{\mathbb{T}_G}.$$

Now in the stable model structure, cofibrations are cofibrations in the level structure, and fibrations are maps satisfying the right lifting property with respect to cofibrations which are (stable) equivalences. One proves similarly as in [20] that this does define a closed model structure.

Unless explicitly mentioned otherwise, by *equivalence* of G-$\mathbb{A}^1$-spectra we shall mean a stable equivalence.

2.3. **Functors.** There are many interesting functors analogous to functors present in topological equivariant stable homotopy theory. They are defined in analogous ways as in the topological situation. We will mention only a few examples which we will need here specifically.

There is a *suspension spectrum functor*

$$\Sigma^\infty_G : G\text{-}\mathbb{A}^1\text{-based spaces} \to G\text{-}\mathbb{A}^1 - \text{spectra},$$

left Quillen adjoint to

$$\Omega^\infty_G : G\text{-}\mathbb{A}^1\text{-spectra} \to G\text{-}\mathbb{A}^1\text{-based spaces}.$$

There is also a "push-forward functor"

$$(?)_{fixed} : \mathbb{A}^1\text{-spectra} \to G - \mathbb{A}^1\text{-spectra}$$

where one puts

$$(E_{fixed})_n := S^{n\widetilde{\mathbb{A}^G}} \wedge E_n$$

where $\widetilde{\mathbb{A}^G}$ is the reduced regular representation of G. (One uses the fact that we have a canonical isomorphism $\widetilde{\mathbb{A}^G} \oplus \mathbb{A}^1 \cong \mathbb{A}^G$.) Then the functor $(?)_{fixed}$ is left Quillen adjoint to the *fixed point functor*

$$(?)^G : G\text{-}\mathbb{A}^1\text{-spectra} \to \mathbb{A}^1\text{-spectra}.$$

Another example of a functor in which we will be interested is, for a based G-$\mathbb{A}^1$-space X and a G-$\mathbb{A}^1$-spectrum E, the G-$\mathbb{A}^1$-spectrum

$$X \wedge E$$

which is given by

$$(X \wedge E)_n := X \wedge E_n,$$

with structure maps induced from those of E.

A particularly interesting case is the case when $X = G_+$. In this case, we can actually also consider the functors

$$F[G, ?), G \rtimes ? : \mathbb{A}^1\text{-based spaces} \to G\text{-}\mathbb{A}^1\text{-based spaces}$$

which are the right and left adjoint, respectively, to the functor $(?)_{\{e\}}$ which forgets G-structure (=the right and left Kan extension). There are also analogous functors with spaces replaced by spectra. We will need

Lemma 3. *The adjunction between $F[G, ?)$ and $(?)_{\{e\}}$ on the level of spaces or spectra is a Quillen adjunction.*

Proof: It is obvious that $(?)_{\{e\}}$ preserves equivalences as well as cofibrations, which implies the statement. □

2.4. **Equivariant motivic symmetric spectra.** The category of G-equivariant motivic symmetric spectra for G finite will be needed in the last section, where we will need a formalism for proving E_∞-ring structure on the motivic real cobordism spectrum. The required category of symmetric spectra is obtained by combining the methods of Mandell [32] and Jardine [20].

One defines a G-equivariant motivic symmetric spectrum X as a G-$\mathbb{A}^1$-spectrum

$$X = (X_n)$$

together with symmetric group actions

$$\Sigma_n \times X_n \to X_n \tag{8}$$

such that the structure map

$$(\mathbb{T}_G)^{\wedge p} \wedge X_n \to X_{p+n}$$

is $(\Sigma_p \times \Sigma_n)$-equivariant. A morphism of G-equivariant motivic symmetric spectra is a morphism of G-equivariant motivic spectra which is equivariant with respect to the symmetric group actions (8).

Following Jardine [20], one defines a stable closed model structure on G-equivariant motivic symmetric spectra as follows: Stable fibrations are simply morphisms which are stable fibrations on the underlying G-equivariant motivic spectra. Stable equivalences are maps $f : X \to Y$ of G-equivariant motivic symmetric spectra where for every W an injective stably fibrant G-equivariant motivic symmetric spectrum,

$$f^* : \underline{hom}(Y, W) \to \underline{hom}(X, W)$$

is an equivalence of simplicial sets. Here an injective fibration is a map which satisfies the right lifting property with respect to all maps which are level-wise cofibrations and level-wise equivalences. An injective object is an object X such that the map $X \to *$ where $*$ is the terminal object is an injective fibration. (This is a precise equivariant analogue of the discussion on p.509 of [20].) Recall here that the simplicial set

$$\underline{hom}(X, Y)$$

is defined by

$$(\underline{hom}(X, Y))_n = hom(X \wedge \Delta^n_+, Y),$$

where *hom* is the ordinary categorical *hom*-set, and Δ^n is the standard simplicial n-simplex. Stable cofibrations are simply maps which satisfy the left lifting property with respect to acyclic stable fibrations.

3. The Wirthmüller and Adams isomorphisms

3.1. **The Wirthmüller isomorphism.**

Theorem 4. *(The Wirthmüller isomorphism) If E is an $\mathbb{A}^1$-spectrum, then there is a natural equivalence*

$$F[G,E) \simeq G \rtimes E. \tag{9}$$

Proof: We will prove that $G \rtimes ?$ is right adjoint to the functor $(?)_{\{e\}}$ in the homotopy category, whence our statement will follow by uniqueness of adjoints. (Note that the functor $(?)_{\{e\}}$ preserves equivalences.)

First note that choosing an embedding

$$G \subset \mathbb{A}^G$$

yields a Pontrjagin-Thom G-map

$$t : \mathbb{T}_G \to \mathbb{A}^G/(\mathbb{A}^G - G) \simeq G_+ \wedge \mathbb{T}_G,$$

in other words,

$$t : \mathbb{T}_G \to G_+ \wedge \mathbb{T}_G,$$

or, stably,

$$t : S^0 \to G_+. \tag{10}$$

We define the unit to be, for a G-$\mathbb{A}^1$-spectrum E,

$$\eta := t \wedge Id : E \to G_+ \wedge E \cong G \rtimes E_{\{e\}}. \tag{11}$$

Let us, for the moment, not worry about whether this functor preserves equivalences; if we define both unit and counit on the "point set level" (i.e. before passage to homotopy categories), and prove the triangle identities in the homotopy categories, this will follow.

Let, then, the counit be defined, for an $\mathbb{A}^1$-spectrum E, as the map

$$\epsilon : (G \rtimes E)_{\{e\}} \to E \tag{12}$$

gotten by noticing that non-equivariantly, $G \rtimes E$ is just a wedge sum of $|G|$ copies of E, and taking Id on the copy corresponding to $e \in G$, and the collapse map to the point on the other copies.

To verify the triangle identities, let us first look at "$R \to RLR \to R$" (where R, L stands for right and left adjoint). One has an isomorphism

$$G \rtimes (G \rtimes E)_{\{e\}} \cong (G \times G) \rtimes E, \tag{13}$$

which allows us to write our composition as

$$G \rtimes E \to G_+ \wedge (G \rtimes E) \cong (G \times G) \rtimes E \to G \rtimes E \tag{14}$$

where the last map is obtained by observing that $(G \times G) \rtimes E$ is a wedge of G copies of $G \rtimes E$, and taking the identity on the copy corresponding to

$e \in G$, and collapsing the other copies to the base point. In these terms, the composition of the first two maps is identified just with a "multplication by $|G|$" map, i.e. with the map (11) interpreted non-equivariantly as a map $S^0 \to |G|_+$, smashed with identity on $G \rtimes E$. We see that the composition of these two maps is the identity.

Let us now consider "$L \to LRL \to L$". Clearly, however, this map is just the composition

$$E_{\{e\}} \to |G|_+ \wedge E_{\{e\}} \to E_{\{e\}}$$

where the first map is the "multiplier map" and the second map is again the map which is identity on the wedge copy corresponding to e, and collapse to the base point on the other copies. Clearly, again, this is homotopic to the identity. □

Corollary 5. *When $f : E \to F$ is an equivalence of G-$\mathbb{A}^1$-spectra, then*

$$X_+ \wedge f : X_+ \wedge E \to X_+ \wedge F$$

is an equivalence of G-$\mathbb{A}^1$-spectra when X is the pushforward of a simplicial G-set $S_.$ where S_n is a free G-set for all n.

Proof: We claim that the case $X = G$ follows directly from Theorem 4. In effect, the Theorem implies that $G \rtimes ?$ preserves equivalences, and we have the natural isomorphism

$$G \rtimes E_{\{e\}} = G_+ \wedge E$$

which proves the statement. Thus, the statement follows by induction on simplicial skeleta, and by preservation of equivalences by direct limits of sequences of cofibrations (which is true for model structures which are left proper [12] - an axiom satisfied for the Nisnievich topology and generalized to the present equivariant context in a straightforward way). □

3.2. **The Adams isomorphism.**

Lemma 6. *Let EG_n be the simplicial n-skeleton of EG. Then there exists a G-set S and an inclusion of vector bundles*

$$\Phi : \mathbb{A}^G \times_G EG_n \to \mathbb{A}^S \times BG_n. \tag{15}$$

Proof: We choose as an equivalent model of $\mathbb{A}^G \times_G EG_n$ the G-$\mathbb{A}^1$-space

$$B((\mathbb{A}^G \times_G EG_n)^\circ, \Delta^{Op}, \mathbb{A}^{\cdot}) \tag{16}$$

where $(?)^\circ$ denotes barycentric subdivision, Δ is the simplicial category (we write the two-sided bar construction so that the first coordinate is covariant and last contravariant) and $\mathbb{A}^\cdot$ is the standard cosimplicial object

$$(\mathbb{A})^n = \{(x_0, ..., x_n) \in \mathbb{A}^{n+1} | \sum x_i = 1\}$$

(the "algebraic model" of the standard simplex cosimplicial object).

Now let

$$S = G^{n+1},$$

$$\lambda(v, g_1, ..., g_k) := (0, ..., 0, v, \underbrace{0, ..., 0}_{k}),$$

and define a map from (16) to $\mathbb{A}^S$ by

$$\phi((w_0, ..., w_k), [s_0, ..., s_k]) := \sum_{i=0}^{k} s_i \lambda(w_i)$$

where $\sigma_0 \supset ... \supset \sigma_k$ are simplices in EG_{n+1} and

$$w_i := v_i \times_G \sigma_i.$$

Then ϕ is the first coordinate of (15), the second coordinate being just the projection to

$$B((BG_n)^\circ, \Delta^{Op}, \mathbb{A}^\cdot).$$

□

Lemma 7. *Let E be a G-$\mathbb{A}^1$-spectrum. Then there exists a natural (stable) equivalence*

$$\psi_n : EG_{n+} \wedge_G E \xrightarrow{\simeq} (EG_{n+} \wedge E)^G.$$

(The source is the simplicial n-skeleton of $B_\wedge(E, G_+, S^0)$.)

Proof: Consider the G-equivariant inclusion

$$EG_n \xrightarrow{\subseteq} \mathbb{A}^G \times_G EG_n \longrightarrow \mathbb{A}^S \times BG_n$$

induced by the natural inclusion

$$G \subset \mathbb{A}^G.$$

Pull back via

$$\tilde{G}_n := B(G, G, G) \to B(G, G, *)_n = EG_n,$$

we get

$$\tilde{G}_n \subset \mathbb{A}^G \times_G \tilde{G}_n \to \mathbb{A}^S \times B(*, G, G)_n \subset S^{\mathbb{A}^S} \wedge B(*, G, G)_{n+}. \tag{17}$$

Factoring out the complement of the image in (17), we get

$$S^{\mathbb{A}^S} \wedge B(*, G, G)_{n+} \to S^{\mathbb{A}^S} \wedge \tilde{G}_n. \tag{18}$$

(Note that the pullback of a trivial vector bundle is trivial.) Applying $? \wedge_G E$ to (18) gives

$$S^{\mathbb{A}^S} \wedge EG_{n+} \wedge_G E \to S^{\mathbb{A}^S} \wedge EG_{n+} \wedge E.$$

Delooping by $S^{\mathbb{A}^S}$, we get

$$EG_{n+} \wedge_G E \to EG_{n+} \wedge E.$$

The source is a pushforward of a fixed spectrum, so $(?)^G$ can be applied to the target. This is, by our definition, ψ_n.

This map is an equivalence, since on the cofiber of the map from the k-skeleton to the $k-1$-skeleton of EG_{n+}, we get a wedge of suspensions of the Wirthmüller isomorphism

$$F(G_+, E_{\{e\}})^G \to (G_+ \wedge E_{\{e\}})^G.$$

□

The map ψ_n depends on n, but clearly remains the same up to homotopy if we replace the map Φ of Lemma 6 by a map homotopic through inclusions of vector bundles. Similarly, we clearly obtain a homotopic map if we replace the set S by $S \subset S'$ without altering the inclusion Φ. Then the usual "Milnor trick" shows that by enlarging S to $S \amalg S$, we can make any two inclusions Φ of Lemma 6 homotopic: On the level of $\mathbb{A}^S$, first apply a linear homotopy moving the first S coordinates to the last, and then a linear homotopy between one choice of Φ using the first S coordinates and another choice of Φ using the last S coordinates. Thus, ψ_{n+1} restricted to the n-skeleton of EG coincides with ψ_n up to homotopy, and we get

Theorem 8. *(The Adams isomorphism) Under the assumptions of Lemma 7, there exists a natural $\mathbb{A}^1$-equivalence*

$$\psi : EG_+ \wedge_G E \xrightarrow{\simeq} (EG_+ \wedge E)^G.$$

□

3.3. **The Tate diagram.** Similarly as in the topological context [11], considering the cofibration

$$EG_+ \to S^0 \to \widetilde{EG},$$

for a G-equivariant motivic spectrum E we now may consider the diagram with rows cofibration sequences:

$$\begin{array}{ccccc} EG_+ \wedge E & \longrightarrow & E & \longrightarrow & \widetilde{EG} \wedge E \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow \\ EG_+ \wedge E & \longrightarrow & F(EG_+, E) & \longrightarrow & \hat{E} \end{array} \tag{19}$$

where

$$\hat{E} = \widetilde{EG} \wedge F(EG_+, E)$$

is the *Tate spectrum* (note that the canonical map $EG_+ \wedge E \to EG_+ \wedge F(EG_+, E)$ is an equivalence by Corollary 5). By the Adams isomorphism, taking G-fixed points, we obtain a diagram with rows cofibration sequences:

$$\begin{array}{ccccc} EG_+ \wedge_G E & \longrightarrow & E^G & \longrightarrow & \Phi^G E \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow \\ EG_+ \wedge_G E & \longrightarrow & F(EG_+, E)^G & \longrightarrow & \hat{E}^G. \end{array} \tag{20}$$

Either diagram (19) or (20) is referred to as the *Tate diagram*.

It is worth commenting on the functor $\Phi^G E$, which, in accordance with the terminology of [30], we call *geometric fixed points*. Noting that

$$(\mathbb{T}_G)^G \cong \mathbb{P}^1, \tag{21}$$

and using arguments similar to [30], we may compute, for a G-equivariant motivic spectrum E given by

$$\mathbb{T}_G \wedge E_n \to E_{n+1}, \tag{22}$$

the geometric fixed points by taking G-fixed points on both sides of (22), including $\mathbb{T}_G$: we let

$$(\Phi^G E)_n = (E_n)^G,$$

and make the structure maps (recall (21))

$$\mathbb{P}^1 \wedge (E_n)^G \to (E_{n+1})^G. \tag{23}$$

4. $\mathbb{Z}/2$-equivariant representability of Hermitian K-theory

4.1. $\mathbb{Z}/2$-equivariant dimensions. For the remainder of the paper, we will focus on $G = \mathbb{Z}/2$. In this section, we will generalize the results of Hornbostel and Hornbostel-Schlichting [14, 15]. Let us first make some remarks on the "dimensions" which occur for $\mathbb{Z}/2$-equivariant motivic spectra. We have an equivariant factorization

$$\mathbb{A}^{\mathbb{Z}/2} = \mathbb{A}^1 \times \mathbb{A}^1_- \tag{24}$$

where $\mathbb{A}^1$ has trivial $\mathbb{Z}/2$-action and $\mathbb{A}^1_-$ has $\mathbb{Z}/2$-action where the generator acts by -1. Next, (24) induces an isomorphism

$$\mathbb{T}_{\mathbb{Z}/2} \cong \mathbb{P}^1 \wedge \mathbb{P}^1_-. \tag{25}$$

We give $\mathbb{P}^1$ the base point ∞. In (25), $\mathbb{P}^1$ has trivial $\mathbb{Z}/2$-action, while on $\mathbb{P}^1_-$, the generator of $\mathbb{Z}/2$ acts by multiplication by -1.

Next, however, we recall that by the basic Nisnievich square [37], $\mathbb{P}^1$ and $\mathbb{P}^1_-$ decompose further. In effect, if we denote by S^1 resp. S^α the simplicial circle resp. $\mathbb{G}_m$ with trivial $\mathbb{Z}/2$-action, we have, as usual,

$$S^1 \wedge S^\alpha \simeq \mathbb{P}^1. \tag{26}$$

Regarding $\mathbb{P}^1_-$, we get from the same diagram

$$S^\gamma \wedge S^{\gamma\alpha} \simeq \mathbb{P}^1_- \tag{27}$$

where S^γ is the (barycentric subdivision of the) simplicial circle with the canonical (=sign) involution, and $S^{\gamma\alpha}$ is $\mathbb{G}_m^{1/z}$, which is $\mathbb{G}_m$ with the involution $z \mapsto 1/z$. (In fact, (27) is more easily seen if we change coordinates on $\mathbb{P}^1_-$ by a fractional linear map to move the fixed points to $1, -1$, which transforms the action so that the generator of $\mathbb{Z}/2$ acts by $z \mapsto 1/z$. The basic Nisnievich square then gives the desired decomposition.)

In any case, we conclude from (26) and (27) that we have a $\mathbb{Z}/2$-equivariant decomposition

$$\mathbb{T}_{\mathbb{Z}/2} \simeq S^{1+\alpha+\gamma+\gamma\alpha} \tag{28}$$

where, as usual, addition in the "exponent" of the sphere indicates smash product.

4.2. **The periodicity theorem.** Let R be a commutative ring with involution, and let M be a finitely generated projective R-module. A *Hermitian form on R* is a bilinear map

$$\omega : M \otimes_{\mathbb{Z}} M \to R$$

which satisfies

$$\omega(ax, y) = a\omega(x, y),\ a \in R,$$
$$\omega(x, ay) = \overline{a}\omega(x, y),\ a \in R,$$
$$\omega(x, y) = \overline{\omega(y, x)}$$

where $\overline{(?)}$ is the involution in R. For a projective R-module M with a Hermitian form ω, we denote by

$$O(M) \tag{29}$$

the group of all automorphisms A of M as an R-module which satisfy

$$\omega(Ax, y) = \omega(x, A^{-1}y). \tag{30}$$

Comment: The notation (29) may seem odd, since thinking of the example of $R = \mathbb{C}$, and $\overline{(?)}$ being complex conjugation, it would seem more appropriate to denote this group as $U(M)$. We should keep in mind, however, that we can also think of (and in fact, the original emphasis was mostly on) fixed rings, in which case the notation (29) seems to make more sense.

Now if we worked in the category of $\mathbb{Z}/2$-equivariant smooth affine schemes, we could denote by

$$K\mathbb{R}_0^{alg} \tag{31}$$

the $\mathbb{Z}/2$-equivariant motivic space which is the fibrant replacement of the sheafification of

$$Spec(R) \mapsto \Omega B(\coprod_M BO(M)) \tag{32}$$

where R is a ring with involution $\overline{?}$, or, in other words, the ring of coefficients of a smooth affine $\mathbb{Z}/2$-scheme, and M is a set of representatives of isomorphism classes of finitely generated projective R-module with a Hermitian form. Note: As stated, the definition may seem not functorial, but we have the usual remedy: following [14], we may define for two finitely generated projective R-modules M, N $O(M, N)$ as the set of isomorphisms $A : M \to N$ satisfying (30); these categories are "functorial" with respect to base change, but not small; picking (small) skeleta using the class axiom of choice gives functoriality for our definition.

However, as noted in [14], it is still not known if Hermitian K-theory is homotopy invariant, in particular, if it satisfies Zariski descent on arbitrary smooth schemes. Therefore, extending this definition to the site of $\mathbb{Z}/2$-equivariant smooth schemes must be handled with care.

Comment: Since this paper was written, M. Schlichting alerted us to two developments: First of all, in his new paper [42], he proves Zariski descent for Hermitian K-theory for arbitrary schemens with an ample family of line bundles. Further, as a consequence, Hermitian K-theory is homotopy invariant on regular Noetherian separated schemes over $\mathbb{Z}[1/2]$. This simplifies the treatment introduced below in this context.

For a Noetherian scheme X, Jouanolou [22], Lemma 1.5 provides a finite-dimensional vector bundle torsor $W \to X$ which is an affine scheme. Following Weibel, Thomason [47], Appendix, we can make this construction functorial. Let us work, say, on the site of smooth separable schemes of finite type over $Spec(k)$. Consider the category C_X whose objects are tuples

$$\alpha = \{W_\alpha \to X, \{W_\alpha|f \to Y,\ f : Y \to X\}\}$$

consisting of a torsor W_α of a finite-dimensional vector bundle over X, and explicit choices of pullbacks of W_α by all maps $Y \to X$ in the site. Morphisms $\alpha \to \beta$ are morphisms of torsors $W_\alpha \to W_\beta$ over X. This defines a strict functor

$$Sm/k \to Cat,$$

and the category assigned to an object V of Sm/k has a small skeleton I_V. Such data give a functor

$$?_{aff} := \prod Obj(I_?) : Sm/k \to \text{affine schemes}, \tag{33}$$

and a natural transformation

$$V_{aff} \to V$$

in the category of schemes over $Spec(k)$. Because this construction is strictly functorial, it is equivariant with respect to finite groups. Of course, we must be careful with applying Hermitian K-theory, since V_{aff} is no longer a smooth scheme. Nevertheless, it is easy to see that (33) is an inverse limit of smooth equivariant schemes: all we need is to consider subproducts over sets of factors which include, with each factor, all images under the finite group in question.

When we pass to coefficient rings, (33) turns into an infinite tensor product, i.e. a direct limit of finite tensor products, and, as remarked above, in the equivariant case, the finite tensor products can be taken to be equivariant. Let us now specialize to the situation of interest to $K\mathbb{R}_0^{alg}$. Because of the fact that V_{aff} is not smooth, we do not want to consider finitely generated projective modules with a Hermitian form over $O_{V_{aff}}$ directly, but pushforwards M of finitely generated projective modules over finite $\mathbb{Z}/2$-equivariant sub-tensor products V'_{aff}: two such modules with Hermitian form will be considered isomorphic if they become isomorphic after pushforward to a larger $\mathbb{Z}/2$-equivariant sub-tensor product V''_{aff}. Such modules over V_{aff} will be referred to as *strictly finite projective modules with a Hermitian form*. Then we can replace (32) by the presheaf of based simplicial sets on $S(G)$ given by

$$V \mapsto \Omega B(\coprod_M BO(M)) \tag{34}$$

where M ranges over representatives of isomorphism classes (in the above sense) of strictly finite $O_{V_{aff}}$-projective modules with Hermitian form M. A key point is the following

Lemma 9. *The canonical map from (34) to (32) is an equivalence of simplicial sets when V is a $\mathbb{Z}/2$-equivariant affine scheme.*

Proof: Clearly, we may express V_{aff} as a directed inverse limit of $\mathbb{Z}/2$-equivariant finite-dimensional vector bundle torsors. For such torsor, we may further cover V by a finite system of $\mathbb{Z}/2$-equivariant Zariski open subsets over each of which the torsor is trivial. Since Hermitian K-theory satisfies Zariski descent in the category of commutative rings with involution, as well as $\mathbb{A}^1$-homotopy invariance (cf. [14]), the statement follows. □

Theorem 10. *We have $\mathbb{A}^1$-equivalences of $\mathbb{Z}/2$-equivariant motivic spaces*

$$\Omega^{\alpha+\gamma} K\mathbb{R}_0^{alg} \simeq K\mathbb{R}_0^{alg}, \quad \textit{Hornbostel-Schlichting } [15] \tag{35}$$

$$\Omega^{1+\gamma\alpha} K\mathbb{R}_0^{alg} \simeq K\mathbb{R}_0^{alg}, \tag{36}$$

$$\Omega^{4} K\mathbb{R}_0^{alg} \simeq \Omega^{4\gamma} K\mathbb{R}_0^{alg}, \quad \textit{Karoubi } [23] \tag{37}$$

Proof: We will first prove (37). Hornbostel [14] (following Karoubi [23] and Kobal [25]) writes down fiber sequences

$$U \longrightarrow F(\mathbb{Z}/2_+, K\mathbb{R}_0^{alg}) \xrightarrow{H} K\mathbb{R}_0^{alg}, \tag{38}$$

$$V \longrightarrow K\mathbb{R}_0^{alg} \xrightarrow{F} F(\mathbb{Z}/2_+, K\mathbb{R}_0^{alg}) \tag{39}$$

where H is "hyperbolization" and F is the forgetful map. He then quotes Karoubi [23] to prove

$$\Omega({}_-U) \simeq V, \quad \Omega(U) \simeq {}_-V \tag{40}$$

where ${}_-(?)$ denotes the analogues of all the above constructions with quadratic forms replaced by symplectic forms. Essentially by definition, we have

$$V \simeq \Omega^{\gamma} K\mathbb{R}_0^{alg}. \tag{41}$$

One next checks that

$$\Omega^{\gamma} U \simeq \Omega K\mathbb{R}_0^{alg}. \tag{42}$$

Indeed, to this end, it suffices to check that the map

$$F(\Sigma^{\gamma}\mathbb{Z}/2_+, K\mathbb{R}_0^{alg}) \to \Omega^{\gamma} K\mathbb{R}_0^{alg} \tag{43}$$

given by $\Omega^{\gamma} H$ is the same as the one induced by the canonical "pinching map" (the Pontrjagin construction)

$$S^{\gamma} \to \mathbb{Z}/2_+ \wedge S^{\gamma}. \tag{44}$$

This can be done directly from the definition. The left hand side of (43) is represented by

$$\Omega B(\coprod_M BGL(M))$$

where M is as in (32). One may in fact deloop once and consider the model of (43) in the form

$$\Omega^\gamma B(\coprod_M BGL(M)) \to \Omega^\gamma B(\coprod_M BO(M)). \tag{45}$$

Using simplicial approximation, the two maps (45) are then readily seen to coincide by definition.

Now by (41), (42),

$$\Omega^{2\gamma} K\mathbb{R}_0^{alg} \simeq \Omega^\gamma V \simeq \Omega^{1+\gamma}(_-U) \simeq \Omega^2(_-K\mathbb{R}_0^{alg}). \tag{46}$$

Similar arguments hold if we add $_-(?)$ everywhere, which gives (37).

Now (35) is essentially Proposition 5.1 of [14], namely that

$$K\mathbb{R}_0^{alg}(R) \to K\mathbb{R}_0^{alg}(R[t,t^{-1}]) \to \Omega^\gamma K\mathbb{R}_0^{alg}(R) \tag{47}$$

is a (split) homotopy fibration for every ring R (in [14], it is stated only for fixed rings, but Theorem 1.8 of [15], which [14] cites, applies to rings with involution as well).

To prove (36), we remark that an analogous argument to [14, 15] also holds with $R[t,t^{-1}]$ replaced by $R^-[t,t^{-1}]$ where involution is given by

$$t \mapsto -t.$$

In effect, to make this precise, we must review some of the concepts of [15]. Let $(A,\overline{?})$ be a commutative ring with involution in which 2 is invertible. Consider an element $f \in A$ which is a non-divisor of 0, such that

$$\overline{f} = -f. \tag{48}$$

Recall from [15], 1.3, that a *category with duality* $(C,\sharp,\eta)$ is a category C with a functor $\sharp : C \to C^{Op}$ and a natural equivalence $\eta : Id_C \to \sharp\sharp$ such that

$$Id_{A^\sharp} = \eta_A^\sharp \circ \eta_{A^\sharp}.$$

The associated Hermitian category is then defined as follows: Objects are pairs (M,ϕ) where M is an object of C and

$$\phi : M \xrightarrow{\cong} M^\sharp$$

is an isomorphism such that $\phi = \phi^\sharp \eta$. A morphism $\alpha : (M, \phi) \to (N, \psi)$ is a morphism $\alpha : M \to N$ in C such that $\alpha^\sharp \psi \alpha = \phi$.

Using an analogue of Quillen's Q-construction, [15], Definition 1.3 defines spaces

$$ {}_\epsilon W(C), \quad {}_\epsilon U(C) = \Omega({}_\epsilon W(C)) $$

generalizing the corresponding concepts for rings, where $\epsilon \in \{\pm 1\}$, and the subscript ϵ indicates replacing C with the category with duality $(C, \sharp, \epsilon\eta)$.

Let Σ be the multiplicative set generated by f. Then in [15], one defines a category with duality T_Σ as follows: Objects are injective morphisms of projective A-modules

$$ i : P_1 \to P_0 \tag{49} $$

which become isomorphisms when we invert Σ. The group of morphisms from (49) to

$$ i' : P'_1 \to P'_0 \tag{50} $$

is the group of commutative squres

$$ \begin{array}{ccc} P_1 & \xrightarrow{i} & P_0 \\ \downarrow & & \downarrow \\ P'_1 & \xrightarrow{i'} & P'_0 \end{array} \tag{51} $$

modulo the subgroup of all squares (51) which split by maps $P_0 \to P'_1$.

The duality is given by the contravariant functor $(?)^\sharp$ where, for an A-module M,

$$ \begin{aligned} M^\sharp &= Hom_{skew}(M, A) \\ &= \{f : M \to A \mid f(am) = \overline{a} f(m) \text{ for all } a \in A\}. \end{aligned} $$

Thus, the dual of (49) is

$$ i^\sharp : P_0^\sharp \to P_1^\sharp. $$

The localization theorem, Theorem 1.8 of [15], then applies directly to our situation, and gives a homotopy fibration

$$ {}_\epsilon U(T_\Sigma) \to {}_\epsilon K\mathbb{R}_0^{alg}(A) \to {}_\epsilon K\mathbb{R}_0^{alg}(A_\Sigma) \tag{52} $$

where A_Σ is the ring A with Σ inverted.

To identify the first term of (52) in the case of $A = R^-[t]$, $\Sigma = \{1, t, t^2, ..., \}$, however, we cannot apply the dévissage theorem, Theorem 1.11 of [15], directly, since that result applies only to the case when $\overline{f} = f$, which differs from (48).

In fact, to extend the method to our case, we must carefully investigate the concept of morphism of categories with duality

$$(C, \sharp, \eta) \to (D, \sharp, \tau).$$

This is a functor

$$F : C \to D$$

together with a natural equivalence

$$\lambda : \sharp F \to F^{Op}\sharp \tag{53}$$

such that the following diagram commutes:

$$\begin{array}{ccccc} & & F & & \\ & \swarrow^{\eta} & & \searrow^{\tau} & \\ \sharp\sharp F & \xleftarrow{\sharp\lambda} & \sharp F^{Op}\sharp & \xleftarrow{\lambda^{-1}\sharp} & F\sharp\sharp. \end{array} \tag{54}$$

(This generalizes slightly the definition of [15], which require an *equality* in (53). In fact, this is precisely what Schlichting [41] calls a *non-singular form functor*. In the present case, we need the generalized definition, which causes no substantial change in the arguments.)

Then, by letting $F(A)$ denote the category of free A-modules, we define, in the situation of (48), a morphism of categories with dualities

$$F : {}_{-\epsilon}F(A) \to {}_{\epsilon}T_{\Sigma} \tag{55}$$

by sending M to

$$M \xrightarrow{f} M. \tag{56}$$

Recalling carefully (48), we let λ be the square

$$\begin{array}{ccc} M^{\sharp} & \xrightarrow{-f} & M^{\sharp} \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle -Id} \\ M^{\sharp} & \xrightarrow{f} & M^{\sharp}. \end{array} \tag{57}$$

We note that in diagram (54), the double dualization will introduce minus signs in both vertical arrows of the comparison square, hence the minus sign in (55). Further, similarly to [15], 1.6, the corresponding square (a morphism version of (56)) becomes split for morphisms which are divisible by f, and hence we obtain a morphism of categories with duality

$${}_{-\epsilon}F(A/fA) \to {}_{\epsilon}T_{\Sigma}. \tag{58}$$

Further, (58) obviously extends to the respective idempotent completions (a technical point needed in (48)), and hence induces a map

$${}_{-\epsilon}W(A/fA) \to {}_{\epsilon}W(T_{\Sigma}). \tag{59}$$

We now have

Theorem 11. *(dévissage): In the present situation, i.e. a commutative ring with involution A in which 2 is invertible, a non-zero divisor f satisfying (48), and the multiplicative set Σ generated by f, the map (59) is an equivalence.*

Proof: Analogous to [15, 10], although a few commets are in order. Hornbostel-Schlichting [15] state a dévissage theorem for rings with involution with the exception that (48) is replaced by

$$\overline{f} = f, \tag{60}$$

and accordingly in (59), the minus sign on the left hand side is deleted. That result is, in effect, needed in the proof of (35) above. The strategy of the proof in [15] is to use Karoubi induction, proving equality between Balmer-Witt groups and classical Witt groups in negative dimension, and quoting [10] for a dévissage theorem for Balmer-Witt groups. The Karoubi induction argument works analogously in our case, in fact, the argument [15] can essentially just be adopted verbatim.

The reference [10], on the other hand, strictly speaking, does not apply to either our present situation or to the case of [15], as [10] only considers fixed rings (i.e. where the involution is the identity). However, studying the method of [10] in detail shows that it can, in effect, be adapted both to our present case and to the case of [15].

To do this, let us first note that in the case of fixed rings, Gille [10] considers a substantially more general context of Gorenstein rings with finite Krull dimension. The basic idea of the proof is to filter by dimension of support, and use a localization spectral sequence [10], 3.3. In the non-equivariant case, this reduces the statement to the case of local Gorenstein rings R where the Balmer-Witt groups with support in the maximal ideal m are proved to be isomorphic, with appropriate shift, to the Witt groups of the residue field. We do not know whether this method generalizes to $\mathbb{Z}/2$-equivariant rings in the generality of Gorenstein rings of finite Krull dimension. The problem is that in the case of local Gorenstein rings, one relies on minimal injective resolutions, the behavior of which under involution we don't fully understand.

However, for our purposes, it suffices to consider *regular* rings. In this case, the $\mathbb{Z}/2$-equivariant analogue of [10], 3.3 leads to two different local cases. When the maximal ideal m is not invariant under the $\mathbb{Z}/2$-action, we are back to the non-equivariant case. In the case when m is invariant under the $\mathbb{Z}/2$-action, we need to show that the Balmer-Witt groups of a

$\mathbb{Z}/2$-equivariant regular local ring R with support in the maximal ideal m are isomorphic, with the appropriate shift, to the Witt groups of the residue field.

More precisely, we have the following. By Luna's slice theorem [31], for a $\mathbb{Z}/2$-equivariant regular local ring R of Krull dimension n with maximal ideal m, we can find n generators

$$m = (f_1, ..., f_n)$$

(called regular parameters) such that

$$\overline{f}_i = -f_i \text{ for } 1 \leq i \leq q,$$

$$\overline{f}_i = f_i \text{ for } q+1 \leq i \leq n$$

for some $q \leq n$. Let k be the residue field. Our statement than reduces to the following analogue of Lemma 4.4 of [10]. □

Lemma 12. *There is a natural diagram of isomorphisms*

$$\begin{array}{ccc} W(k) & \longrightarrow & {}_{((-1)^q)}\tilde{W}^n_m(R) \\ & \searrow & \uparrow \\ & & {}_{((-1)^{q+\epsilon})}\tilde{W}^n_{m/f_i}(R/f_i) \end{array} \tag{61}$$

where $\epsilon = 1$ if $i \leq q$ and $\epsilon = 0$ otherwise, and the vertical map is the canonical one.

Remarks: $\tilde{W}^n_m(R)$ means the obvious extension of Balmer-Witt groups with support [10], Definition 2.16, to rings with involution. A minus sign in front of Balmer-Witt groups on the right hand side of (61) indicates shift of the number n by 2 (the groups are 4-periodic).

Proof: We will only consider the top row of the diagram. The naturality contained in the diagram will follow from the construction. In the present regular case, we may use projective rather than injective resolutions. Let $D^b(\mathcal{P}_{fg,m}(R))$ denote the bounded derived category of complexes of projective R-modules with finitely generated homology with support in m. Define then a functor

$$\iota : D^b(\mathcal{P}(k)) \to D^b(\mathcal{P}_{fg,m}(R)) \tag{62}$$

which sends k to the complex

$$\bigotimes_{i=1}^{n} \left(R \xrightarrow{f_i} R \right) \tag{63}$$

where the target of each of the morphisms is set in dimension 0, and the tensor product is over R. The behavior of (62) with respect to duality is analogous to the analysis we made above. On the right hand side, we can take the duality $Hom_R(?, R)$, which however has to be shifted by n. Further, one must be careful in choosing the signs in the duality isomorphism on (63). We may choose the sign to be, say, $(-1)^k$ on the R-term which has dimension 1 in precisely k factors $1 \leq i \leq q$ in (63). Since the dual switches dimension of each factor between 0 and 1, we see that the duality isomorphism on the right hand side of (62) must be multiplied by the sign $(-1)^q$.

What is left is showing that the map (62) induces isomorphism of Balmer-Witt groups, which is, in effect, our final reduction of the dévissage theorem. To this end, we consider the diagram

$$\begin{array}{ccc}
D^b(\mathcal{P}(k)) & & \\
\downarrow \iota & \searrow^{\iota'} & \\
D^b(\mathcal{P}_{fg,m}(R)) & \longleftarrow & D^b(\mathcal{P}_{fg,m,semis.}(R)) \\
a \downarrow \sim & & \sim \downarrow c \\
D^b(\mathcal{M}_{fg,m}(R)) & \longleftarrow & D^b(\mathcal{M}_{fg,m,semis.}(R)) \\
b \uparrow \sim & & \sim \uparrow d \\
D^b(\mathcal{M}_{fl}(R)) & \underset{e}{\longleftarrow} & D^b(\mathcal{M}_{fl,semis.}(R))
\end{array}$$

Here $D^b(\mathcal{M}_{fg,m}(R))$ denotes the bounded derived category of chain complexes of R-modules with finitely generated homology with support in m, and the symbols $D^b(\mathcal{P}_{fg,m,semis.}(R))$, $D^b(\mathcal{M}_{fg,m,semis.}(R))$ mean full subcategories on complexes whose homology is semisimple. The categories in the last row mean derived categories of the abelian categories $\mathcal{M}_{fl}(R)$ of modules of finite length, and $\mathcal{M}_{fl,semis.}(R)$ of semisimple modules of finite length. Comparisons of dualities in the spirit of [10], Theorem 3.9 have to be made, but no additional signs arise here. Now ι' is an equivalence of categories, as are a, c (by regularity) and b, d (by direct inspection). All equivalences of categories which preserve duality induce isomorphism on Balmer-Witt groups (e.g. Theorem 2.7 of [10]). Thus, it suffices to show that the map e induces an isomorphism on Balmer-Witt groups. To this end, one first notes that the corresponding Balmer-Witt groups are isomorphic to the classical Witt groups of the underlying abelian categories (Balmer [2]). Then, one uses the "Jordan-Hölder theorem" of Quebbemann, Scharlau, Schulte [38]. □

By the Theorem, setting $A = R^-[t]$, $f = t$, we obtain in our situation a homotopy fibration

$$(64) \qquad {}_{-\epsilon}K\mathbb{R}_0^{alg}(R) \to {}_{\epsilon}K\mathbb{R}_0^{alg}(R^-[t,t^{-1}]) \to \Omega^{\gamma}_{\epsilon}K\mathbb{R}_0^{alg}(R).$$

But now note that if we put

$$\mathbb{G}_m^- := SpecR^-[t,t^{-1}].$$

we have a cofibration

$$(65) \qquad (\mathbb{G}_m^-)_+ \to S^0 \to \mathbb{P}^1_-.$$

Recalling (27), the last term of (65) is

$$S^{\gamma+\gamma\alpha}.$$

Thus, replacing $R[t,t^{-1}]$ by $R^-[t,t^{-1}]$ in (47) yields

$$\tilde{K}_0^{alg}(\mathbb{P}^1_-) \simeq {}_-\tilde{K}_0^{alg}(\mathbb{P}^1),$$

i.e.

$$\Omega^{\gamma+\gamma\alpha}K\mathbb{R}_0^{alg} \simeq \Omega^{1+\alpha}{}_-K\mathbb{R}_0^{alg},$$

so (36) follows from (35). □

4.3. The $\mathbb{Z}/2$-equivariant motivic spectrum $K\mathbb{R}^{alg}$.

Comment: From (36), it follows that if we denote by $R'[t,t^{-1}]$ the ring $R[t,t^{-1}]$ with involution $t \mapsto 1/t$, we get a split cofibration

$$(66) \qquad K\mathbb{R}_0^{alg}R \to K\mathbb{R}_0^{alg}(R'[t,t^{-1}]) \to \Omega K\mathbb{R}_0^{alg}R,$$

which answers a question implicit in [14], the paragraph before 5.1.

The periodicity theorem now implies

$$(67) \qquad K\mathbb{R}_0^{alg} \simeq \Omega^{1+\alpha+\gamma+\gamma\alpha}K\mathbb{R}_0^{alg} \simeq F(\mathbb{T}_{\mathbb{Z}/2}, K\mathbb{R}_0^{alg}).$$

Thus, the standard method gives a $\mathbb{Z}/2$-equivariant motivic spectrum $K\mathbb{R}^{alg}$ whose 0-space is $K\mathbb{R}_0^{alg}$, and which satisfies

$$(68) \qquad \mathbb{T}_{\mathbb{Z}/2} \wedge K\mathbb{R}^{alg} \simeq K\mathbb{R}^{alg}.$$

5. The completion theorem

5.1. The "Karoubi tower". Let

$$c : S^0 \to S^\gamma$$

be the canonical inclusion of fixed points.

Theorem 13. *The equivalence (35) can be chosen in such a way that the composition*

$$S^0 \xrightarrow{c} \Sigma^\gamma K\mathbb{R}^{alg} \xrightarrow{\iota} \Sigma^{-\alpha} K\mathbb{R}^{alg}$$

is homotopic to η.

Proof: The composition

$$\mathbb{Z}/2_+ \longrightarrow S^0 \xrightarrow{\eta} \Sigma^{-\alpha} K\mathbb{R}^{alg}$$

is 0 since $\eta = 0 \in \pi_\alpha K^{alg}$ (a formula true for any algebraically oriented motivic spectrum, cf. [18]), which gives the top square of a diagram

$$\begin{array}{ccc}
\mathbb{Z}/2_+ & \xrightarrow{0} & \Sigma^{-\alpha} K\mathbb{R}^{alg} \wedge \mathbb{Z}/2_+ \\
\downarrow & & \downarrow \lambda \\
S^0 & \xrightarrow{\eta} & \Sigma^{-\alpha} K\mathbb{R}^{alg} \\
c \downarrow & \nearrow \iota & \downarrow c \\
S^\gamma & \overset{\kappa}{\dashrightarrow} & \Sigma^{-\alpha-\gamma} K\mathbb{R}^{alg} \\
\downarrow & & \downarrow \\
\Sigma\mathbb{Z}/2_+ & \xrightarrow{0} & \Sigma^{1-\alpha} K\mathbb{R}^{alg} \wedge \mathbb{Z}/2_+ .
\end{array} \tag{69}$$

The maps κ, ι exist for formal reasons. Thus, we have

$$\eta = \iota c \tag{70}$$

for some ι. Now we use the commutative ring structure on $K\mathbb{R}^{alg}$ (defined by the standard methods analogous to other kinds of K-theory, i.e. tensor product of bundles, etc.) to reinterpret (69) as a diagram

$$\begin{array}{ccc}
K\mathbb{R}^{alg} & \xrightarrow{\eta} & \Sigma^{-\alpha} K\mathbb{R}^{alg} \\
c \downarrow & \nearrow \iota & \\
\Sigma^\gamma K\mathbb{R}^{alg} . & &
\end{array} \tag{71}$$

Let W resp. GW denote the Witt resp. Grothendieck-Witt ring of the base field. Taking $\pi_{m\alpha}$, $m > 1$ on (71), the diagram becomes a diagram of maps of W-modules

$$\begin{array}{ccc} W & \xrightarrow{\eta} & W \\ {\scriptstyle c}\downarrow & \nearrow_{\iota} & \\ W. & & \end{array} \tag{72}$$

Further, η is an isomorphism (multiplication by a unit), and hence so are c, ι. (Recall that $K\mathbb{R}^{alg}_{\alpha+\gamma} = K\mathbb{R}^{alg}_0 = GW$ by (35).) We therefore know that the reduction of the map $\iota \in GW$ to W is a unit. Now note that on π_γ, the map in (70) becomes the inclusion

$$\mathbb{Z} \xrightarrow{[H]} GW.$$

Therefore, the proof of the Theorem is concluded by the following result.□

Lemma 14. *Let $\alpha \in GW$ be an element which reduces to a unit in W. Then there exists an $m \in \mathbb{Z}$ such that $\alpha + m[H]$ is a unit in GW.*

Proof: We have a $\beta \in GW$ such that

$$\alpha\beta = 1 + n[H].$$

Taking augmentation, we get

$$ab = 1 + 2n,$$

so the integers a, b must be odd, say,

$$a = 2k + 1,$$

$$b = 2\ell + 1.$$

We compute

$$\begin{aligned}
&(\alpha - k[H])(\beta - \ell[H]) = \\
&\alpha\beta - (2k+1)\ell[H] - (2\ell+1)k[H] + 2k\ell[H] = \\
&\alpha\beta - (2k\ell + k + \ell)[H] = \\
&\alpha\beta - n[H] = 1.
\end{aligned}$$

□

5.2. **The completion problem for Hermitian K-theory.** The Tate diagram for $K\mathbb{R}^{alg}$ (after taking fixed points) looks as follows:

$$
\begin{array}{ccccc}
K\mathbb{R}^{alg}_{h\mathbb{Z}/2} & \longrightarrow & KH & \longrightarrow & KT \\
{\scriptstyle =}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle s}\downarrow \\
K\mathbb{R}^{alg}_{h\mathbb{Z}/2} & \longrightarrow & (K\mathbb{R}^{alg})^{h\mathbb{Z}/2} & \longrightarrow & (\widehat{K\mathbb{R}}^{alg})^{\mathbb{Z}/2}.
\end{array}
\tag{73}
$$

Hence the top cofibration sequence is the one constructed in Kobal [25], where KH denotes (affinized) Hermitian K-theory and KT is Balmer-Witt K-theory [14]. (Note however that we are working with "affinized" versions of all the theories in question.)

One may also suspend by any dimension $k + \ell\alpha + m\gamma + n\gamma\alpha$ before taking fixed points in (73). However, by the periodicities proved in Theorem 10 above, only suspensions by $k + \ell\alpha$ give new information, and they are already contained in (73).

The completion problem asks in general in what sense (if any) the middle vertical arrow of (73) is a completion, or becomes an equivalence after completion. To address this question, first recall [11] that we may write the basic cofibration sequence

$$E\mathbb{Z}/2_+ \to S^0 \to \widetilde{E\mathbb{Z}/2}$$

as the homotopy direct limit of

$$S(m\gamma)_+ \longrightarrow S^0 \xrightarrow{c^m} S^{m\gamma},$$

so the middle vertical arrow of the Tate diagram before taking fixed points

$$E \to F(E\mathbb{Z}/2_+, E)$$

can be identified with the canonical map

$$E \to \operatorname{holim}_{\overleftarrow{n}} E/c^n.$$

Similarly, we may write

$$\widehat{E} = c^{-1} \operatorname{holim}_{\overleftarrow{n}} E/c^n, \quad \widetilde{E\mathbb{Z}/2} \wedge E = c^{-1}E.$$

Using Theorem 13 above, we therefore deduce

Theorem 15. *There are natural $\mathbb{Z}/2$-equivariant equivalences*

$$\operatorname{holim}_{\overleftarrow{n}} K\mathbb{R}^{alg}/\eta^n \simeq F(E\mathbb{Z}/2_+, K\mathbb{R}^{alg}), \tag{74}$$

(75) $$\eta^{-1}\ \underset{\overleftarrow{n}}{holim}\ K\mathbb{R}^{alg}/\eta^n \simeq \widehat{K\mathbb{R}^{alg}}.$$

□

Note that η is an element in non-equivariant motivic stable homotopy groups, and therefore (74) and (75) can also be stated on the level of fixed points:

$$\underset{\overleftarrow{n}}{\mathrm{holim}}\ ((K\mathbb{R}^{alg})^{\mathbb{Z}/2})/\eta^n \simeq (K\mathbb{R}^{alg})^{h\mathbb{Z}/2},$$

$$\eta^{-1}\ \underset{\overleftarrow{n}}{\mathrm{holim}}\ ((K\mathbb{R}^{alg})^{\mathbb{Z}/2})/\eta^n \simeq (\widehat{K\mathbb{R}^{alg}})^{\mathbb{Z}/2}.$$

However, we may attempt to go further and calculate the homotopy cofiber of the canonical map

(76) $$K\mathbb{R}^{alg} \xrightarrow{q} \underset{\overleftarrow{n}}{\mathrm{holim}}\ K\mathbb{R}^{alg}/\eta^n,$$

which by the Tate diagram is the same as the cofiber of

(77) $$\eta^{-1}K\mathbb{R}^{alg} \to \eta^{-1}\ \underset{\overleftarrow{n}}{\mathrm{holim}}\ K\mathbb{R}^{alg}/\eta^n.$$

The behavior of the theory $\widehat{K\mathbb{R}^{alg}}$ is described in the following result.

Theorem 16. *The* $(* + *\alpha + *\gamma + *\gamma\alpha)$*-graded coefficients of* $\widehat{K\mathbb{R}^{alg}}$ *have periodicities* γ, α, 4, $1+\gamma\alpha$. *Further,*

(1) For $n \not\equiv 0 \mod 4$, *we have* $(\widehat{K\mathbb{R}^{alg}})_n = 0$.

(2) The map $(\Phi^{\mathbb{Z}/2}K\mathbb{R}^{alg})_0 = KT_0 \to (\widehat{K\mathbb{R}^{alg}})_0$ *is the map*

(78) $$W \to \underset{\overleftarrow{n}}{\lim}\, W/I^n$$

where I is the augmentation ideal of the Witt ring W.

Proof: First, note that Tate-cohomology is always γ-periodic, and the periodicities stated are a formal consequence of that and the periodicities of Real algebraic K-theory proved in Theorem 10.

The main idea of the argument proving (1) and (2) is to calculate $\widehat{K\mathbb{R}^{alg}}$ by a "slice spectral sequence". We shall however not discuss an analogue of Voevodsky's theory of slices [46] for G-equivariant spectra in the present paper. Instead, we observe that Borel and Tate cohomology can be calculated in the category of *naive* G-equivariant motivic spectra, by which we mean ordinary (non-equivariant) $\mathbb{P}^1$-spectra equipped with a (strict) G-action. $K\mathbb{R}^{alg}$-theory can be represented in this category by the presheaf on the category of affine smooth schemes over $Spec(k)$ which sends $Spec(R)$ (for a non-equivariant ring R) to

$$\Omega B(\coprod_M BGL(M))$$

where M is a finitely generated projective R-module with a quadratic form, the action is by $A \mapsto (A^T)^{-1}$, where T is adjunction with respect to the quadratic form.

To see that this is the right construction, recall the remarks in Subsection 2.1 on "forgetting equivariant structure" on a $\mathbb{Z}/2$-equivariant motivic space X: one applies the functor to schemes of the form $\mathbb{Z}/2\times ?$. In our case, $\mathbb{Z}/2 \times Spec(R)$ is $Spec(R \prod R)$, where the $\mathbb{Z}/2$-equivariant structure can be taken as interchanging the factors, so if a $Spec(R \prod R)$-module M is obtained by change of basis from an R-module N, then we have $O(M) \cong GL(N)$.

The construction of the coniveau tower due to Levine [28, 29] is functorial, and thus applies automatically to the category of naive G-spectra. More specifically, Levine [29] defines, for a motivic spectrum E, a functorial *homotopy coniveau tower*

$$... \rightarrow E^{(p+1)} \rightarrow E^{(p)} \rightarrow E^{(p-1)} \rightarrow ... \tag{79}$$

whose homotopy (inverse) limit is E, which realizes Voevodsky's slice tower [46]. Levine [28, 29] showed that the slices of ordinary algebraic K^{alg} are $H\mathbb{Z}^{Mot}$. The (unrigidified) $\mathbb{Z}/2$-action on the slice

$$H\mathbb{Z}^{alg} \rightarrow H\mathbb{Z}^{alg} \tag{80}$$

can be identified by comparison with the topological case [17]; in dimensions where no suspension by γ or $\gamma\alpha$ is present, the action is trivial; v_1 induces a periodicity. Applying the construction (79) to a naive G-spectrum E, we obtain a double tower, indexed in n, p

$$F_G(EG_{n+}, E^{(p)}) \tag{81}$$

whose homotopy limit is the Borel cohomology of E (recall that by EG_n we mean the n-skeleton of the reduced bar construction $B(G, G, *)$). Therefore, "totalizing this (co)-filtration" in any way we choose, we obtain a spectral sequence conditionally convergent to the Borel cohomology of E. By inverting the pushforward of the one point compactification of the simplicial model of the reduced regular representation of G (the direct limit of iterated smash products of this space is $\widetilde{EG}$), we obtain a spectral sequence conditionally convergent to $\widehat{E}_*$.

In our present case $G = \mathbb{Z}/2$, $E = K\mathbb{R}^{alg}$, we find advantageous the totalization of the degrees (81) which equates, in filtration degree, one slice of the Levine tower with two cell dimensions of $E\mathbb{Z}/2$. Specifically, let

$$K\mathbb{R}^{alg}_{<<n>>}$$

be the homotopy fiber of

$$\begin{aligned} &\prod_{k,q} \prod_{m+j-\ell=n} \Sigma^{k\gamma+q\alpha+j\gamma\alpha} F_G((EG_{k-2m})_+, E^{(-\ell)}) \\ (82) \qquad &\rightrightarrows \prod_{k,q} \prod_{m+j-\ell=n-1} \Sigma^{k\gamma+q\alpha+j\gamma\alpha} F_G((EG_{k-2m})_+, E^{(-\ell)}) \end{aligned}$$

where the maps are obtained by lowering either of the indices m, ℓ by 1.

In the case of the corresponding Tate spectral sequence, which we by convention ([17]) grade homologically, the fixed point spectrum of the associated graded object is a product of copies of the smash product of the Moore spectrum $M\mathbb{Z}/2$ with $H\mathbb{Z}^{Mot}$ (by (80), the connecting map of the 2-cell free $\mathbb{Z}/2$-CW complexes into which we have cut $E\mathbb{Z}/2$, when smashed with $H\mathbb{Z}^{Mot}$, is 2); this smash product is $H\mathbb{Z}/2^{Mot}$, whose coefficients we know by the Milnor conjecture, proved by Voevodsky [45]. For instance, when $j = k = \ell = 0$, the term on the left hand side of (82) involving the $2m$-skeleton of $E\mathbb{Z}/2$ is in filtration degree $-m$. If we increase k by 1, (which is where the canonical element $c : S^0 \to S^\gamma$ is present), the $2m+1$-skeleton will end up in filtration $-m$, (and similarly linearly in k). In the 1-st slice, everything is periodic by multiplying by v_1, which increases j and ℓ by 1 (and again, similarly in multiples). This is the reason we chose the filtration in the way specified above. Accounting for all the copies of $H\mathbb{Z}/2$, we get

$$\begin{aligned} E^1 &= H\mathbb{Z}/2^{Mot}_*[\lambda, \lambda^{-1}][\sigma^2, \sigma^{-2}][c, c^{-1}][v_1, v_1^{-1}] \\ &= (K_M(F)/2)_*[\theta][\lambda, \lambda^{-1}][\sigma^2, \sigma^{-2}][c, c^{-1}][v_1, v_1^{-1}], \end{aligned}$$

where the dimensions of elements are given by

$$|\sigma^2| = 2 - 2\gamma,\ |c| = -\gamma,\ |v_1| = 1 + \gamma\alpha,\ |\lambda| = 1 + \gamma\alpha - \gamma - \alpha, |\theta| = 1 - \alpha,$$

and the filtration degrees of all these elements are 0 except

$$deg(\sigma^{2k}) = k$$

(we list all the dimensions, since with this filtration, the spectral sequence isn't really a spectral sequence of rings). But now comparing with the topological case (over the field $\mathbb{C}$, see [17]), we get

$$d^1(\sigma^{4k+2}) = v_1 c^3 \theta \lambda^{-1} \sigma^{4k},$$

so

$$E_2 = (K_M(F)/2)_*[\lambda, \lambda^{-1}][\sigma^4, \sigma^{-4}][c, c^{-1}][v_1, v_1^{-1}]$$
$$= E_0(W_I^\wedge)[\lambda, \lambda^{-1}][\sigma^4, \sigma^{-4}][c, c^{-1}][v_1, v_1^{-1}].$$

The map from $KT = \Phi^{\mathbb{Z}/2}K\mathbb{R}^{alg}$ proves that all of these elements are permanent cycles, as claimed. □

We observe that by Corollary 5.2 (p.352) of [26],

$$\bigcap_{i=0}^{\infty} I^n = 0, \tag{83}$$

and thus the map (78) is always injective.

On the other hand, it is also immediate that (78) is an isomorphism if and only if there exists an n such that $I^n = 0$, which is not true in general:

Proposition 17. *If F is any field of characteristic 0, adjoining infinitely many transcendental variables $x_1, ..., x_n, ...$, then the field $k = F(x_1, ..., x_n, ...)$ satisfies*

$$I^n \neq 0 \tag{84}$$

for all n.

Proof: We may consider the inclusion

$$k \subset k[\sqrt{x_1}, ..., \sqrt{x_n}]. \tag{85}$$

This is a Galois extension, with the Galois group a product of n copies of $\mathbb{Z}/2$. Thus, the mod 2 Galois cohomology of k maps to the Galois cohomology with $\mathbb{Z}/2$ coefficients of (85), which is

$$\Lambda_{\mathbb{F}_2}(a_1, ..., a_n),$$

with x_i mapping to a_i. Thus, we see that the symbol $[x_1, ..., x_n]$ is non-zero in $K^M/2(k)$, which, by the Milnor conjecture (proved by Voevodsky [45]) implies (84). □

Thus, for the choice of k of Proposition 17, the map (76) is *not* an equivalence. One may next ask ([6, 25], etc.)

(86) Does the map (77) become an equivalence after completion at 2?

Here the notion of completion at 2 might seem ambiguous, since it could mean localization $L_{M\mathbb{Z}/2}$ at $M\mathbb{Z}/2$ or $\operatorname{holim}_{\overleftarrow{n}} (?)/2^n$. Recall, however, the following result:

Lemma 18. *There is a canonical equivalence*

$$L_{M\mathbb{Z}/2}E \to \underset{\overleftarrow{n}}{holim}\ E/2^n. \tag{87}$$

Proof: (following Bousfield [7], but reproduced to emphasize its independence, to a large degree, of the model structure): First, note that we have a canonical map

$$E \to \underset{\leftarrow}{holim}\, E/(2^n), \tag{88}$$

and that this map induces equivalence after smashing with $M\mathbb{Z}/2$: because of stability, smashing with $M\mathbb{Z}/2$ commutes past the *holim*, so on the right hand side of (88) we have

$$\underset{\leftarrow}{holim}(E/(2^n) \wedge M\mathbb{Z}/2), \tag{89}$$

but the content of the parentheses is

$$E/2 \vee \Sigma E/2, \tag{90}$$

and the structure map of the homotopy limit is 0 on the second factor (this follows from writing explicitly the cofibration sequence with respect to multiplying by 2 and then, on the result, multiplying by 2^n).

Thus, (89), which is again the right hand side of (88) divided by 2 is $E/2$, and clearly (88) divided by 2 is the identity (since in each constituent of the holim, we get the identity to the first summand of (90)).

Thus, (88) is an equivalence after smashing with $M\mathbb{Z}/2$, so it suffices to prove that the right hand side of (88) is $M\mathbb{Z}/2$-local. This means that for every spectrum Y with

$$Y \wedge M\mathbb{Z}/2 = 0, \tag{91}$$

the mapping spectrum from Y to the right hand side of (88) is 0. Clearly, such a property however is preserved by the holim, and for each constituent, this follows again from the fact that (91) is equivalent to $2 : Y \to Y$ being an equivalence. □

Let us also note that $L_{M\mathbb{Z}/2}$ preserves cofibration sequences, and $E = KT$ is "completable" in the sense that it satisfies

$$Hom(\mathbb{Z}/2^\infty, \pi_* E) = 0 \tag{92}$$

by (83), and the fact that $2 \in I$. We also immediately see then that the answer to question (86) is *no* in general: if F is any field which contains $\sqrt{-1}$, then $2 = 0 \in W_F$. Thus, completion at 2 is the identity on the coefficients of KT. On the other hand, putting $F := \mathbb{Q}(i)$, we see by Proposition 17 that for the field $F(x_1, ..., x_n, ...)$, the completion at I is non-trivial on the field k defined there.

This further suggests restricting quesion (86) to fields of finite transcendence degree. In this case, we can, indeed, prove that the map (78) becomes iso on coefficients after 2-completion. In fact, we can prove a more general statement. We already remarked that $2 \in I$.

Lemma 19. *For any field k with $cd_2 k[i] = n < \infty$, and for any $N > m \geq n+1$,*

$$2I^m + I^N = I^{m+1}. \tag{93}$$

Proof: The statement is trivial when $i \in k$. Thus, let us assume $i \notin k$. First consider the Serre spectral sequence in mod 2 Galois cohomology for the field extension $k \subseteq k[i]$. We have

$$E_2 = H^*_{Gal}(k[i], \mathbb{Z}/2)[([-1])] \tag{94}$$

where $[-1]$ is the generator of $H^*(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2[([-1])]$, and $[-1]$ has bidegree $(p, q) = (1, 0)$. Further $[-1]$ is a permanent cycle in (94), so one proves by induction that

$$[-1] : E_r^{p,q} \to E_r^{p+1,q} \tag{95}$$

is an isomorphism for $p \geq r$. However, note that $E_{n+1} = E_\infty$ for dimensional reasons. Thus, we have shown that

$$[-1] : H^m_{Gal}(k, \mathbb{Z}/2) \to H^{m+1}_{Gal}(k, \mathbb{Z}/2) \tag{96}$$

is an isomorphism for $m \geq n+1$. By the Milnor conjecture, we can then replace $H^m_{Gal}(?, \mathbb{Z}/2)$ by $K^m_M(?)/2$, which however is the associated graded object of W with respect to filtration by powers of the augmentation ideal I. Noting further that $2 \in W$ is represented by $[-1] \in K^1_M(?)/2$, (93) is just a restatement of this fact. □

We note that the book Elman, Karpenko, Merkurjev [8] contains several related statements, but we could not find the precise statement of Lemma 19.

Theorem 20. *The answer to the question (86) is yes on coefficients over a point for fields k satisfying $cd_2(k[i]) < \infty$.*

Proof: By Theorem 16 and the subsequent discussion, it suffices to show that the canonical map

$$\lim_{\substack{\leftarrow \\ n}} W/2^n \to \lim_{\substack{\leftarrow \\ n}} W/I^n \tag{97}$$

is an isomorphism. We shall use Corollary 5.2 (3) (p.353) of Lam [26], which asserts that

$$\bigcap_n (\phi \cdot W + I^n) = \phi \cdot W \tag{98}$$

for any Pfister form ϕ. Since $2^k \in W$ is representable by a Pfister form, Lemma 19 implies that for every m, there exists an n such that

$$I^n \subseteq 2^m.$$

Thus, powers of the ideals 2 and I induce the same uniformity, and our statement follows. □

Comment: We note that an analogue of the Theorem for $p > 2$ in the spirit of [19] is also true, although less interesting. When $cd_p(F) < \infty$, the completion of W at p is 0 (since W is 2-torsion), while its completion at the augmentation ideal is 2-complete, and hence the p-completion of (77) is an isomorphism, both sides being 0.

6. Motivic Real cobordism

6.1. **The construction of motivic Real cobordism.** Consider the hyperbolic quadratic form on k^{2n}:

$$q(x_1, ..., x_{2n}) = x_1x_2 + ... + x_{2n-1}x_{2n}. \tag{99}$$

The associated symmetric bilinear form is

$$b((x_1, ..., x_{2n}), (y_1, ..., y_{2n})) = \sum_{i=1}^{n} x_{2i}y_{2i-1} + x_{2i-1}y_{2i}. \tag{100}$$

The b-adjoint of a matrix $A = (a_{ij})_{i,j=1}^n$ is an $n \times n$ matrix A^{T_b} such that

$$b(Ax, y) = b(x, A^{T_b}y). \tag{101}$$

Explicitly, putting $A^{T_b} = (b_{ij})_{i,j=1}^n$, one has

$$b_{2i,2j} = a_{2j-1,2i-1}, \tag{102}$$

$$b_{2i-1,2j-1} = a_{2j,2i}, \tag{103}$$

$$b_{2i,2j-1} = a_{2j,2i-1}, \tag{104}$$

$$b_{2i-1,2j} = a_{2j-1,2i}. \tag{105}$$

There is an involution on the algebraic group GL_{2n} given by

$$A \mapsto (A^{T_b})^{-1}. \tag{106}$$

Note that then the resulting group $\mathbb{Z}2 \ltimes GL_{2n}$ acts on the quadric

$$Q_b^t := V(x, y | b(x, y) = t), \ \ t \in k^\times \tag{107}$$

where $V(x_i|E)$ (sometimes further abbreviated $V(E)$) denotes the locus of the equations E in the variables x_i, and the involution on (107) is

$$x \leftrightarrow y \tag{108}$$

Recall that Q_b^t has the non-equivariant $\mathbb{A}^1$-homotopy type of

$$\mathbb{A}^{2n} - \{0\} \simeq S^{(2n-1)+2n\alpha}. \tag{109}$$

A non-equivariant $\mathbb{A}^1$-equivalence from (107) to (109) is the projection

$$(x, y) \mapsto x.$$

Lemma 21. *There is a $\mathbb{Z}/2$-equivariant isomorphism*

$$Q_b^1(x, y) \to Q_b^t(x', y'), t \in k^\times$$

given by

$$\begin{array}{ll} x'_{2i} = tx_{2i}, & y'_{2i} = ty_{2i}, \\ x'_{2i-1} = x_{2i-1}, & y'_{2i-1} = y_{2i-1}. \end{array} \tag{110}$$

Furthermore, this isomorphism becomes $\mathbb{Z}/2 \ltimes GL_{2n}$-equivariant, with respect to an isomorphism

$$\psi : GL_{2n} \to GL_{2n}, \ (a_{ij}) \mapsto (a'_{ij})$$

where

$$\begin{array}{ll} a'_{2i,2j} = a_{2i,2j}, & a'_{2i,2j-1} = ta_{2i,2j-1} \\ a'_{2i-1,2j-1} = a_{2i-1,2j-1}, & a'_{2i-1,2j} = t^{-1}a_{2i-1,2j}. \end{array} \tag{111}$$

Proof: A direct computation. □

Let us now define the *join* $X * Y$ of G-$\mathbb{A}^1$-spaces X, Y as the colimit of the diagram

$$\begin{array}{ccccc} X \times Y \times \{0\} & \longrightarrow & X \times Y \times \mathbb{A}^1 & \longleftarrow & X \times Y \times \{1\} \\ \downarrow & & & & \downarrow \\ X \times \{0\} & & & & Y \times \{1\} \end{array}$$

where the horizontal arrows are inclusions, and the vertical ones are projections. Define further the *unreduced suspension* $\overline{X}$ of G-$\mathbb{A}^1$-space X as the colimit of the diagram

$$\begin{array}{ccccc} X \times \{0\} & \longrightarrow & X \times \mathbb{A}^1 & \longleftarrow & X \times \{1\} \\ \downarrow & & & & \downarrow \\ * & & & & *. \end{array}$$

It is possible to show that $*$ defines a symmetric monoidal structure. The join of n objects $X_1, ..., X_n$ can be canonically identified with the coend of

$$X_S \times_C \mathbb{A}_S \tag{112}$$

where C is the category of non-empty subsets of $\{1, ..., n\}$ and inclusions,

$$X_S = \prod_{i \in S} X_i$$

is a contravariant functor by projection, and

$$A_S = V(x_1, ..., x_n | \sum x_i = 1, x_i = 0 \text{ for } i \notin S) \tag{113}$$

is a covariant functor by inclusion.

We would like to claim that

$$\overline{X * Y} \cong \overline{X} \wedge \overline{Y}. \tag{114}$$

Unfortunately, this is false. A partial remedy can be obtained as follows. Let $\tilde{X}$ be the functorial fibrant replacement of $\overline{X}$. We can then construct a contractible operad $\mathcal{D}$ and a natural equivalence

$$\mathcal{D}(n)_+ \wedge \tilde{X}_1 \wedge ... \wedge \tilde{X}_n \to X_1 \widetilde{* ... *} X_n \tag{115}$$

which satisfies the obvious operad action diagrams (associativity, unitality and equivariance). The operad $\mathcal{D}$ can be constructed as follows. Consider the diagram of $\mathbb{A}^1$-spaces

$$\mathcal{A}(n) = (\overline{\mathbb{A}_S})_S$$

over $S \subset \{1, ..., n\}$ (see (113)). The arrows of the diagrams are given by inclusions of the sets S. Let

$$\mathcal{B}(n) \tag{116}$$

be the fibrant replacement of $\mathcal{A}(n)$ in the corresponding diagram category of based $\mathbb{A}^1$-spaces. Then we can consider

$$\mathcal{B}(1) \wedge ... \wedge \mathcal{B}(1) \tag{117}$$

as an object in the same category of diagrams, where $(?)_S$ is

$$(\mathcal{B}(1))_{\epsilon_1} \wedge ... \wedge (\mathcal{B}(1))_{\epsilon_n}$$

where ϵ_i is 1 or 0 depending on whether $i \in S$ or not. We can then let $\mathcal{D}(n)$ be the $\mathbb{A}^1$-space of maps from the diagram $\mathbb{A}^1$-space (116) to the diagram $\mathbb{A}^1$-space (117).

Now write $Q_b = Q_b^1$. Writing more specifically b_n instead of b for the bilinear form on the space k^{2n}, we have by Lemma 21 a canonical map

$$Q_{b_{n_1}} * ... * Q_{b_{n_k}} \to Q_{b_{n_1+...+n_k}}, \tag{118}$$

which satisfies the obvious commutative, associative and unital properties; the map is obtained by applying the morphism $Q_{b_{n_i}}$ to $Q^{t_i}_{b_{n_i}}$ where t_i are the coordinates of the join (112) (note that the map is defined and $\mathbb{Z}/2$-equivariant even in the case $t_i = 0$), and we are using the obvious inclusion

(119) $$Q^{t_1}_{b_{n_1}} \times ... \times Q^{t_k}_{b_{n_k}} \subseteq Q_{b_{n_1+...+n_k}} \text{ for } t_1 + ... + t_k = 1.$$

Therefore, if we denote

$$S(n) := \widetilde{Q_{b_n}},$$

then we get canonical maps

(120) $$\mathcal{D}(n)_+ \wedge S(n_1) \wedge ... \wedge S(n_k) \to S(n_1 + ... + n_k)$$

which satisfies the obvious commutativity, associativity and unitality properties.

We will deal with the operad $\mathcal{D}$ later. For now, we need an analogue of the construction (120) to Thom spaces. That is complicated by the fact that the isomorphism (111) is not defined for $t = 0$, so the map (118) defined via (119) cannot be made $GL_{2n_1} \times ... \times GL_{2n_k}$-equivariant by twisting the group action on the target by the isomorphisms ψ of Lemma 21.

Fortunately, the quadrics Q^0_b are contractible (they are cones), so they may be collapsed to a point without altering the homotopy type. More precisely, we do the following. To simplify notation, write

$$u_i = x_{2i}y_{2i-1} + x_{2i-1}y_{2i},$$

$$v_j = u_{n_1+...+n_{j-1}+1} + ... + u_{n_1+...+n_j}.$$

Let us write

$$Q(n) = Q^1_{b_n}.$$

Now denote by $Q'(n_1, ..., n_k)$ the sheaf obtained from $Q(n)$ by collapsing, by projection, for non-empty subsets

$$S \subseteq \{1, ..., k\}$$

the subschemes

(121) $$V(v_j = 0 \text{ for } j \in S, \sum_{j \notin S} v_j = 1)$$

to

(122) $$* \times V(\sum_{i \notin S} v_j = 1)$$

(i.e. we mean all the constituent variables of every v_j, $j \in S$, are omitted). Note that some justification is needed to make the construction, since the subschemes (121) are not regular. We procede in the usual way, i.e. choosing $GL_{2n_1} \times ... \times GL_{2n_k}$-equivariant resolution of singularities, and then collapsing the inverse images. (Obviously, this can be done in the present situation.)

Lemma 22. *The natural map*

$$Q(n) \to Q'(n_1, ..., n_k) \tag{123}$$

is a $\mathbb{Z}/2 \ltimes (GL_{2n_1} \times ... \times GL_{2n_k})$*-equivariant equivalence.*

□

Set

$$S(n) = \widetilde{Q(n)},$$
$$T'(n_1, ..., n_k) = B_\wedge(S^0, (GL_{2n_1} \times ... \times GL_{2n_k})_+, Q'(\widetilde{n_1, ..., n_k})),$$
$$T(n_1, ..., n_k) = B_\wedge(S^0, (GL_{2n_1} \times ... \times GL_{2n_k})_+, \widetilde{Q(n)}).$$

Then we have $\mathbb{Z}/2$-equivariant maps

(124)
$$\mathcal{D}(p)_+ \wedge T'(n_{11}, ..., n_{1q_1}) \wedge ... \wedge T'(n_{p1}, ..., n_{pq_p}) \xrightarrow{\simeq} T'(n_{11}, ..., n_{pq_p}),$$

$$T(n_1, ..., n_k) \xrightarrow{\simeq} T'(n_1, ..., n_k), \tag{125}$$

$$T(n_{11}, ..., n_{pq_1}) \to T(n_{11} + ... + n_{1q_1}, ..., n_{p1} + ... + n_{pq_p}). \tag{126}$$

The maps (124), (125), (126) are unital, associative and equivariant. Here equivariance means with respect to wreaths of symmetric groups which preserve the notation with all possible reindexings. For example, in (124), the general element of the equivariance group is a wreath of a permutation of p elements with the wreaths of permutations of q_1, ..., q_p elements, with the permutations of n_{11},...,n_{pq_p} elements, etc.

Additionally, we have $\mathbb{Z}/2$-equivariant maps

$$S(n_1 + ... + n_k) \to T(n_1, ..., n_k) \tag{127}$$

which satisfy pemutation equivariance, and compatibility with all the structure. The diagram worth mentioning explicitly is

(128)
$$\begin{array}{ccc}
\mathcal{D}(p)_+ \wedge S(n_1) \wedge ... \wedge S(n_p) & \longrightarrow & S(n_1 + ... + n_k) \\
\downarrow & & \downarrow \\
\mathcal{D}(k)_+ \wedge T'(n_{11}, ..., n_{1q_1}) \wedge ... \wedge T'(n_{p1}, ..., n_{pq_p}) & \xrightarrow{\simeq} & T'(n_{11}, ..., n_{pq_p})
\end{array}$$

where

$$n_i = n_{i1} + ...n_{iq_i},$$

which involves (120), (124).

We are now ready to get rid of the operad $\mathcal{D}$. Indeed, this can be done formally as follows. Consider the structure on the objects $T(?)$, $T'(?)$, $S(?)$ specified by (124), (125), (126), (127), (128) (and all the implicit coherence diagrams we did not spell out). Let $M_{\mathcal{D}}$ denote the monad defining such structure, and let M_* denote the monad defining the same structures with $\mathcal{D}$ replaced by the operad $*$ where $*(n) = *$. Then the bar construction

$$B(M_*, M_{\mathcal{D}}, ?)$$

converts our structure to one where $\mathcal{D}$ is trivial, i.e.

$$\mathcal{D}(n) = *. \tag{129}$$

Remark 23. *We should remark here one important difference between our case and the situation, say, of May* [33]. *In* [33], *the map of monads* $M_{\mathcal{D}} \to M_*$ *would not be an equivalence, since the construction of the monad involves factoring the space* $\mathcal{D}(n)$ *by the action of the symmetric group* Σ_n. *When dealing with symmetric objects, however, the symmetric group action is a part of the structure, and hence, in effect, the construction of the monad does not involve this factorization. Hence, the monads preserve equivalence of operads (by which we mean a map of operads which is an equivalence space-wise).*

Hence, we may assume (129) without loss of generality.

We will next show that we may further "rectify" to produce an algebra over the monad M_* with the additional property that

(130) the map (125) is an isomorphism.

Before showing how to accomplish that, let us comment on the significance. Note that if (130) holds, then we simply have $\mathbb{Z}/2$-equivariant maps

$$T(n_1) \wedge ... \wedge T(n_k) \to T(n_1 + ... + n_k) \tag{131}$$

which are associative, unital and equivariant with respect to all wreaths of k permutations of $n_1, ..., n_k$ elements, along with a $\mathbb{Z}/2 \times \Sigma_n$-equivariant map

$$S(n) \to T(n) \tag{132}$$

together with a commutative diagram

$$\begin{array}{ccc} S(n_1) \wedge ... \wedge S(n_k) & \longrightarrow & S(n_1 + ... + n_k) \\ \downarrow & & \downarrow \\ T(n_1) \wedge ... \wedge T(n_k) & \longrightarrow & T(n_1 + ... + n_k) \end{array} \tag{133}$$

Note that (131), (132), (133) define a symmetric monoid in the category of $\mathbb{Z}/2$-equivariant $S(1)$-symmetric spectra. $S(1)$ is a model of $\mathbb{T}_{\mathbb{Z}/2}$. This object can be converted to a $\mathbb{Z}/2$-equivariant $\mathbb{T}_{\mathbb{Z}/2}$-$E_\infty$ ring spectrum by standard categorical manipulations. We will not give the details here. We merely record

Theorem 24. *The above construction produces a $\mathbb{Z}/2$-equivariant motivic E_∞ ring spectrum, which we denote by $MGL\mathbb{R}$.*

Comment: 1. In analogy with the fact that the geometric fixed point spectrum of Landweber's Real cobordism is the unoriented cobordism spectrum, by applying the geometric fixed point functor (Subsection 3.3), we may define a (non-equivariant) motivic spectrum

$$MGLO := \Phi^{\mathbb{Z}/2} MGL\mathbb{R}. \tag{134}$$

Defining this analogue answers a question of Jack Morava.

2. There is also a purely topological application of our construction. There is certainly a topological realization of our definition over the field $F = \mathbb{C}$, which can be shown to give a $\mathbb{Z}/2$-equivariant spectrum equivalent to the spectrum $M\mathbb{R}$ of [17]. On the other hand, our construction for $F = \mathbb{R}$ also has a topological realization, which is properly viewed as a $\mathbb{Z}/2 \times \mathbb{Z}/2$-equivariant spectrum over the complete universe, with underlying non-equivariant spectrum MU. The two $\mathbb{Z}/2$-generators act on matrices by $A \mapsto \overline{A}$, $A \mapsto (A^T)^{-1}$, respectively. (Note that this is not the same thing, since we consider adjunction with respect to the *hyperbolic* form.) We denote this spectrum by $M\mathfrak{L}$, in analogy with Karoubi's $\mathfrak{L}$-theory [24], (recall [14], Appendix, that this is not the same as the L-theory spectrum used in surgery theory, which, for rings which contain $1/2$, is equal to KT), which is viewed properly as a $\mathbb{Z}/2 \times \mathbb{Z}/2$-equivariant spectrum, indexed over the complete universe, with underlying non-equivariant spectrum K. We will investigate the spectrum $M\mathfrak{L}$ in another paper.

Note that to finish the proof of the Theorem, we still need to describe a construction which converts (125) into isomorphisms. This is accomplished by a variant of a construction known as *May-Thomason rectification*. We consider two categories. Recall that we are assuming $\mathcal{D}(n) = *$. We work $\mathbb{Z}/2$-equivariantly throughout. A category $\mathcal{K}$ is the category of tuples T, T' with maps (124), (125), (127). These maps are required to satisfy the relevant permutation equivariances, and unitality and symmetry in the case of (124), and the diagram (128). A subcategory $\mathcal{L}$ consists of all such structures where (125) is an isomorphism. Then we have two functors

$$R : \mathcal{L} \to \mathcal{K}, \quad L : \mathcal{K} \to \mathcal{L} \tag{135}$$

where R is right adjoint to L. In effect, R is the inclusion, and L is the functor which replaces T with T'.

Then we have also monads $M_{\mathcal{K}}$, $M_{\mathcal{L}}$ in the categories $\mathcal{K}$, $\mathcal{L}$ respectively which define the structures with the additional structure map (126), satisfying all the requirements stated above. One sees that these monads preserve equivalences, so one has an equivalence

$$M_{\mathcal{K}} \to RM_{\mathcal{L}}L. \tag{136}$$

Again, (see Remark 23 above), in contrast with the situation [35, 34], we are not required to index the operations (126) by an E_∞-operad, because, essentially, the monads associated with operads in the context of symmetric objects do not involve factoring through the action of Σ_n. The required rectification functor from $\mathcal{K}$ to $\mathcal{L}$ is then the two-sided bar construction of monads

$$B(M_{\mathcal{L}}L, M_{\mathcal{K}}, ?). \tag{137}$$

6.2. **Real algebraic orientation, formal group laws, and the Real motivic spectra series.** In this subsection, we would like to mention some extremely powerful implications of Theorem 24. Essentially, we can now construct motivic analogues of all the "Real" spectra constructed in [17]. First, we develop the notion of a *Real-orientation* of a $\mathbb{Z}/2$-equivariant motivic spectrum. Recall the $\mathbb{Z}/2$-equivariant algebraic group $\mathbb{G}_m^{1/z}$ defined in Subsection 4.1 above. Then we have a natural inclusion

$$\iota : S^{1+\gamma\alpha} \simeq \Sigma\mathbb{G}_m^{1/z} \to B\mathbb{G}_m^{1/z}. \tag{138}$$

Naively, it may seem appropriate to define Real-oriented motivic spectra as $\mathbb{Z}/2$-equivariant motivic commutative associative ring spectrum (not necessarily in any rigid sense) E such that $1 \in E_0$ is in the image of the map

$$\iota^* : \tilde{E}^{1+\gamma\alpha}B\mathbb{G}_m^{1/z} \to \tilde{E}^{1+\gamma\alpha}S^{1+\gamma\alpha}.$$

When this condition is satisfied, call E a $\mathbb{G}_m^{1/z}$*-oriented* $\mathbb{Z}/2$*-equivariant motivic ring spectrum.*

Proposition 25. *When E is a $\mathbb{G}_m^{1/z}$-oriented $\mathbb{Z}/2$-equivariant motivic spectrum, then $E_{*(1+\gamma\alpha)}$ is a commutative ring.*

Proof: (a variation of Lemma 2.17 of [17]). We must show that in the coefficients E_*, the map

$$\epsilon : \mathbb{G}_m^{1/z} \to \mathbb{G}_m^{1/z} \tag{139}$$

given by $z \mapsto 1/z$ induces multiplication by -1. However, the point is that taking the unreduced suspension of (139), by Real orientability, the map into coefficients will factor through

$$\pi_{1+\gamma\alpha}B\mathbb{G}_m^{1/z}. \tag{140}$$

On (140), we have two mutually distributive unital group structures, one coming from the homotopy group, one from the multiplication on $\mathbb{G}_m^{1/z}$. By the standard argument, they must coincide. Now ϵ resp. -1 are the inverses of the element given by the canonical inclusion in the two group structures. □

Proposition 26. *When E is a $\mathbb{G}_m^{1/z}$-oriented $\mathbb{Z}/2$-equivariant motivic spectrum, then*

$$E^*B\mathbb{G}_m^{1/z} = E^*[[u]] \tag{141}$$

where u is the class obtained from the definition of Real orientation. Additionally, the multiplication on $B\mathbb{G}_m^{1/z}$ induces a formal group law on the commutative ring

$$E_{*(1+\gamma\alpha)}.$$

Proof: This is precisely analogous to the proof of the corresponding statement in [17]. □

On the other hand, with this definition, we don't know how to construct Chern classes, or prove universality of $MGL\mathbb{R}$ (in fact, we don't even know that $MGL\mathbb{R}$ itself satisfies the condition).

The reason for this difficulty is, roughly speaking, that our theory has a $\mathbb{Z}^4$-grading: intuitively, in a well behaved definition, the α and γ-graded parts of the theory should also make an appearance. From this point of view, it is more reasonable to consider the following condition:

(142) The unit class in $\widetilde{E}^{1+\gamma+\alpha+\gamma\alpha}S(1)$ extends to a class $w_E \in \widetilde{E}^{1+\gamma+\alpha+\gamma\alpha}T(1)$.

(Note that since BGL_2 is connected in the $\mathbb{Z}/2$-equivariant motivic sense, there is a canonical (up to $\mathbb{A}^1$-homotopy) "fiber" inclusion $S(1) \subset T(1)$.)

The trouble is, however, that we do not know if the condition (142) implies $\mathbb{G}_m$-orientability. What we do have, is a canonical map in the $\mathbb{Z}/2$-equivariant stable motivic homotopy category

(143)
$$S^{2+2\gamma\alpha} \simeq \widetilde{\mathbb{G}_m^{1/z}} \wedge \widetilde{\mathbb{G}_m^{1/z}} \to \widetilde{\mathbb{G}_m^{1/z}} \times \widetilde{\mathbb{G}_m^{1/z}} \to B(\mathbb{G}_m^{1/z} \times \mathbb{G}_m^{1/z}) \to BGL_2 \to T(1).$$

(The last arrow is the 0-section.) Composing (143) with the cohomology class w_E, we obtain an element

$$\lambda_E \in \pi_{1+\gamma\alpha-\gamma-\alpha}E. \tag{144}$$

Definition 27. *We call a $\mathbb{Z}/2$-equivariant motivic (not necessarily strictly) commutative associative unital ring spectrum* Real-oriented *if it satisfies the condition (142), and if the class λ_E of (144) is invertible as an element of the coefficient ring.*

Example: The $\mathbb{Z}/2$-equivariant motivic spectrum $MGL\mathbb{R}$ clearly satisfies the condition (142). It follows that the $\mathbb{Z}/2$-equivariant motivic spectrum $MGL\mathbb{R}[\lambda^{-1}]$ (which can be constructed as an E_∞-ring spectrum by the methods of [9]) is real-oriented. We do not know if the $\mathbb{Z}/2$-equivariant motivic spectrum $MGL\mathbb{R}$ is real-oriented.

Now by Proposition 26, there exists a canonical map

$$L \to MGL\mathbb{R}_*[\lambda^{-1}] \tag{145}$$

where L is the Lazard ring, and in the standard grading of the Lazard ring, an element of degree $2k$ is carried by (145) to an element of degree $k(1+\gamma\alpha)$. Now since $MGL\mathbb{R}$ is additionally an E_∞-ring spectrum, we may apply the constructions of [9], in particular "kill" or "invert" any sequence of elements in L in the spectrum $MGL\mathbb{R}[\lambda^{-1}]$. In analogy with similar spectra in [17], we have in particular a Real algebraic Brown-Peterson spectrum $BP\mathbb{R}$, Real algebraic Johnson-Wilson spectra $BP\mathbb{R}\langle n\rangle^{alg}$, the algebraic $E\mathbb{R}$-theories $E\mathbb{R}(n)^{alg}$, and algebraic Real Morava K-theories $K\mathbb{R}(n)^{alg}$ (these occur one prime at a time, with most interest, as always, in the prime 2).

Remark: Finally, it is worth remarking that using the method of Hill, Hopkins and Ravenel [13], in certain cases, the motivic Real cobordism spectrum can be used to construct, in a completely geometric way, examples of (homotopy) fixed point spectra with respect finite subgroups of Morava stabilizer groups larger than $\mathbb{Z}/2$. While the precise role of such objects in motivic stable homotopy theory is not yet known, in view of the recent paper of Behrens and Hopkins [4], such spectra may be considered a first step on a long road toward the conjectured motivic analogues of topological automorphic form spectra [3]. The point is that the construction [3] of topological automorphic forms relies heavily on Lurie's machinery, which in turn seems to need calculational input currently not available in the motivic case.

Proposition 28. *The spectrum $K\mathbb{R}^{alg}$ is Real-oriented.*

Proof: In effect, this amounts to proving the following result, which is also of independent interest as a geometric construction of some of the periodicity maps of Theorem 10. □

Lemma 29. *The canonical inclusions $SL_2 \to GL_\infty$ resp. $\mathbb{G}_m^{1/z} \to GL_\infty$ (here, as before, we consider hyperbolic involution on SL_2, GL_∞), viewed as elements of $\widetilde{K\mathbb{R}^{alg}}(S^{1+\gamma+\alpha+\gamma\alpha})$ resp. $\widetilde{K\mathbb{R}^{alg}}(S^{1+\gamma\alpha})$ are invertible elements in $\widetilde{K\mathbb{R}^{alg}}_*$.*

Proof: We need to prove that multiplications by the specified elements are isomorphisms in $K\mathbb{R}^{alg}$-cohomology. By a trick of Max Karoubi's ([23], Lemma 2.4, Proposition 2.5, and p. 276), it suffices to prove this statement with $K\mathbb{R}^{alg}$ replaced by L-theory, or topological Hermitian K-theory over $\mathbb{R}$. This theory is the "topological realization" of $K\mathbb{R}^{alg}$ for $F = \mathbb{R}$, and can be viewed as a $\mathbb{Z}/2 \times \mathbb{Z}/2$-equivariant spectrum over the complete universe, whose 0-space is homotopically equivalent to $BU \times \mathbb{Z}$, and the two $\mathbb{Z}/2$-generators act on matrices by $A \mapsto \overline{A}$ and $A \mapsto (A^T)^{-1}$, respectively. (Again, note that the actions do not coincide, since the adjunction is with respect to the hyperbolic form; in some sense, therefore, L-theory combines the information of both real and $\mathbb{Z}/2$-equivariant K-theory.) The periodicity of this theory is treated by Max Karoubi in [24], part III. While the $\mathbb{Z}^4$-graded indexing is not discussed in [24] and this periodicity is left as an exercise (Proposition 3.3), the statement amounts to observing that the representation given in the statement of our Lemma define irreducible Clifford modules of the given signatures (in the case of SL_2, the "equivariant K-theory" $\mathbb{Z}/2$-generator acts by minus on one of the coordinates). This follows, nevertheless, from the well known fact that increasing signature by $(1, 1)$ or $(2, 2)$ corresponds to tensoring the Clifford algebra with an algebra of matrices (again, the other $\mathbb{Z}/2$-generator acts by minus on one of the coordinates in the $(2, 2)$-case). □

We will next prove universality of $MGL\mathbb{R}[\lambda^{-1}]$ among Real-oriented motivic spectra. We will need a couple of preliminary lemmas. First, let us consider the bilinear form

$$b(x, y) = x_1y_1 - x_2y_2 + ... \pm x_ny_n \tag{146}$$

(we continue using the convention $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, and the involution $x_i \leftrightarrow y_i$; the signs in (146) alternate). Let, with this notation,

$$Q_n := Q_b^1.$$

Then we have, in particular,

$$Q_{2n} \cong Q_{b_n} = Q(n). \tag{147}$$

Consider also, from now on, GL_n with involution $A \mapsto (A^T)^{-1}$, where the transposition T is with respect to the form (146). Note that, as usual, in this notation, GL_n acts equivariantly on Q_n.

Lemma 30. *The action of GL_n on Q_n is transitive, and the stabilizer of the point $x^0 = (0,...,0,1)$, $y^0 = (0,...,0,\pm 1)$ is $GL_{n-1} \subset GL_n$ (by inclusion of the first $n-1$ coordinates.*

Proof: The only non-trivial statement is the transitivity. Clearly, GL_n moves any point on Q_n to a point (x,y) where $x = (0,...,0,1)$. Then we must have $y_n = \pm 1$ (the sign being determined by the parity of n). If we set

$$A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ ... & ... & ... & ... \\ & & 1 & \\ a_1 & ... & a_{n-1} & 1 \end{pmatrix}.$$

Then

$$(A^T)^{-1} = \begin{pmatrix} 1 & & & -a_1 \\ & 1 & & +a_2 \\ ... & ... & ... & ... \\ & & 1 & \mp a_{n-1} \\ & ... & & 1 \end{pmatrix}.$$

Thus, we see that

$$A\begin{pmatrix} 0 \\ ... \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ ... \\ 0 \\ 1 \end{pmatrix}, \ (A^T)^{-1}\begin{pmatrix} y_1 \\ ... \\ y_{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ ... \\ 0 \\ 1 \end{pmatrix}$$

when $a_i = (-1)^{i-1} y_i$. □

Lemma 31. *The stabilizer group inclusion from Lemma 30 induces an equivalence*

$$B(*, GL_{n-1}, *) \to B(*, GL_n, Q_n).$$

Proof: We use the fact that Q_n is covered by Zariski-open sets U_i such that if we denote by $p_n : GL_n \to Q_n$ the projection

$$A \mapsto A(x^0, y^0),$$

then

$$(p_n^{-1} U_i \to U_i) \cong (U_i \times GL_{n-1} \to U_i).$$

We may put $U_i = \{(x,y) \in Q_n | x_i y_i \neq 0\}$ (the assertion is proved by the same method as Lemma 30). □

Theorem 32. *Let E be a real-oriented $\mathbb{Z}/2$-equivariant motivic spectrum. Then*

1. We have

$$E^*(B\mathbb{G}_m^{1/z} \times ... \times B\mathbb{G}_m^{1/z}) = E^*[[t_1, ..., t_n]], t_k \in E^{1+\gamma\alpha} B\mathbb{G}_m^{1/z} \tag{148}$$

and

$$E^*(BGL_n) = E^*[[c_1, ..., c_n]],\ c_k \in E^{k+k\gamma\alpha}(BGL_n) \tag{149}$$

such that the canonical inclusion $\mathbb{G}_m^{1/z} \times ... \times \mathbb{G}_m^{1/z} \subset GL_n$ maps c_k to the k-th elementary symmetric polynomial $\sigma_k(t_1, ..., t_n)$.

2. There exists a map of (non-strict) ring spectra

$$MGL\mathbb{R}[\lambda^{-1}] \to E \tag{150}$$

which induces the real orientation on E.

Proof: First note that 1. implies 2. This is because by Lemma 31, we have a cofibration sequence

$$BGL_{2n-1} \to BGL_{2n} \to T(n) \tag{151}$$

where the first map corresponds to the inclusion of the first $2n - 1$ coordinates. By 1., in E^*-cohomology, (151) induces a short exact sequence which we know explicitly. c_{2n} is in the kernel, and gives a "Thom class"

$$\Sigma^{-n(1+\gamma+\alpha+\gamma\alpha)} T(n) \to E. \tag{152}$$

Also by 1., these maps are compatible (up to homotopy) under the structure maps of $MGL\mathbb{R}$ (and also under the ring structure), so passing to the homotopy direct limit over n gives a ring map

$$MGL\mathbb{R} \to E.$$

This factors into (150) because we assume λ is invertible in E.

To prove 1., first note that we may factor (143) through

$$S^{1+\gamma\alpha} \wedge B\mathbb{G}_m^{1/z} \to T(1),$$

which gives a $\mathbb{G}_m^{1/z}$-orientation, which proves (148) by Proposition 26, as well as (149) for $n = 1$.

The challenge in proving (149) for general n is that the usual tools (such as Schubert cells) do not appear to be equivariant under the $\mathbb{Z}/2$-involution. Our main tool is the observation that the direct limit

$$\operatorname{hol}\underset{\substack{\longrightarrow \\ n}}{\operatorname{im}}\ Q_n * .... * Q_n \tag{153}$$

is contractible, and we may obtain a spectral sequence in E^*-cohomology by filtering

$$B(*, GL_n, \operatorname{holim}_{\overrightarrow{n}} Q_n * .... * Q_n) \simeq B(*, GL_n, *) \tag{154}$$

by the number of factors of the join:

$$F_k := F_k(B(*, GL_n, \operatorname{holim}_{\overrightarrow{n}} Q_n * ... * Q_n)) = B(*, GL_n, \underbrace{Q_n * ... * Q_n}_{k+1 \text{ factors}}). \tag{155}$$

Note further that by Lemma 31,

$$F_0 \simeq BGL_{n-1}. \tag{156}$$

In fact, more generally, thinking of

$$B(*, GL_n, Q_n) \to B(*, GL_n, *)$$

as a "sphere bundle", and taking the "induced bundle" ξ via the inclusion corresponding to the first $n-1$ coordinates

$$GL_{n-1} \subset GL_n,$$

we can then interpret F_k/F_{k-1} as the "Thom space" of the k-fold Whitney sum

$$\xi \oplus ... \oplus \xi.$$

Now this bundle is "E-orientable" via the inclusion

$$GL_{n-1} \times GL_1 \to GL_n$$

(and the assumption of λ being invertible in E_*), so using this we may deduce that the E^*-spectral sequence associated with (155) (which one can show to be a spectral sequence of E^*-algebras) has

$$E_1 = E^* BGL_{n-1}[c_n]. \tag{157}$$

Thus, we want to prove our statement by induction, showing that the spectral sequence collapses to E_1.

To this end, we use (148) and comparison with the corresponding spectral sequence with GL_n replaced by

$$\underbrace{\mathbb{G}_m^{1/z} \times ... \times \mathbb{G}_m^{1/z}}_{n \text{ times}}.$$

One proceeds in the same way, and shows that this spectral sequence, to which (157) maps, has

$$E_1 = (E^*[[t_1, ..., t_n]]/(t_1 \cdot ... \cdot t_n))[c_n]. \tag{158}$$

By the induction hypothesis, the map from (157) to (158) is an injection, while (158) collapses by (148). Thus, (157) collapses, concluding the induction step. □

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