Homotopical Representation Theory

Note Title 2/18/2016

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Representation theory of Lie algebras / S.

Foundations: EKMM

<u>Knot theory</u>

Jones Polynomial $\longrightarrow$ Khovanov homology (categorification)

$\langle \emptyset \rangle = 1$

$\langle \bigcirc L \rangle = (q+q^{-1})\langle L \rangle$

$\langle \times \rangle = \langle \asymp \rangle - q\langle)(\rangle$

sl_2

D. Bar Natan

$\langle \emptyset \rangle = 0 \to \mathbb{Z} \to 0$

$\langle \bigcirc L \rangle = V \otimes \langle L \rangle$

$\langle \times \rangle =$

$(0 \to \langle \asymp \rangle \xrightarrow{d} \langle)(\rangle [1] \to 0)$

homotopy types from Morse thy

Cohen, Jones, Segal

Lipshitz Sarkar (2011)

Hu, Kriz, K. : TQFT (embedded in S^3) Khovanov homotopy type

Lawson, Lipshitz, Sarkar : equivalence of both constructions

HOMFLY polynomial $\xrightarrow{\text{Sussan (2009)}}$ sl_k Khovanov homology

representation thy of sl_k

Murakami, Ohtsuki, Yamada

H, K, S

$\Lambda^{a}V \otimes \Lambda^{b}V \longrightarrow \Lambda^{a+b}V$

$\mathcal{C}$

$\mathfrak{sl}_2$ Khovanov invariants type.

V = standard rep. of $\mathfrak{sl}_2$

$K(\mathcal{C}) \otimes \mathbb{C} = \Lambda^{i_1}V \otimes \cdots \otimes \Lambda^{i_j}V$

functors

bounded finite representations of Harish-Chandra pairs $(\mathfrak{gl}_n, GL_{i_1} \times \cdots \times GL_{i_j})$

$i_1 + \cdots + i_j = n.$

over $\mathbb{C}$

(generalized) Bernstein-Gelfand-Gelfand

To be over S, you have to discuss repr. theory of Harish-Chandra pairs $/S$.

Representation thy $/S$:

① Lie algebras

M. Ching:

has contraction

partitions of $\{1, \dots, n\}$

$$\partial(n) = \frac{BK(n)}{\lambda_0 \leq \cdots \leq \lambda_n}$$

these are not the smallest and largest partition, respectively.

$$\boxed{F(\Delta(n), S) = \mathrm{Lie}}$$

Representations: Lie operad modules.

<u>Theorem</u>:

$$\begin{tikzcd} \mathrm{Lie} \arrow[r] \arrow[d] \arrow[dr, phantom, "\sim"] & \mathcal{E}_1 \arrow[d] & \leftarrow \text{Associative} \\ * \arrow[r] & \mathcal{E}_\infty & \leftarrow E_\infty\text{-operad.} \end{tikzcd}$$

② <u>Stability</u> In chain complexes

$$\mathcal{E}(n) \otimes X \otimes \cdots \otimes X \to X$$

$$\underbrace{\mathcal{E}(n)[1-n]} \otimes X[1] \otimes \cdots \otimes X[1] \longrightarrow X[1]$$

What about over S?

$$F(\mathcal{A}(n), S) \simeq S^{1-n}$$

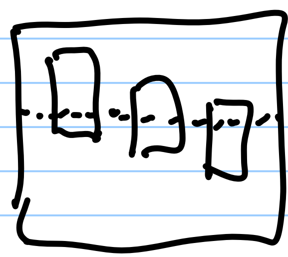
Arone - Kankaanrinta : $\mathcal{A}(n) \not\simeq S^{n-1}$

S-modules

$$\begin{tikzcd}
\text{operads in } \arrow[d, shift right=2, "\underbrace{? \wedge F(\mathcal{A}(n), S)}_{\Sigma^{-1}}"', "\sim"] & \mathcal{S} \arrow[d, "\sim"', "\Sigma"] \\
\text{operads in } \arrow[u, shift right=2, "\Sigma"'] & \mathcal{S}
\end{tikzcd}$$

③ Another model of Ching operad:

$$\mathcal{C}_k \longrightarrow \Sigma \mathcal{C}_{k-1}$$



<u>Theorem</u>: Lie $\sim \underset{\leftarrow}{\operatorname{holim}} \; \Sigma^{1-k} \mathcal{C}_k$.

④ <u>Projection formula</u>:

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i"] \arrow[d, "p"'] \arrow[dr, phantom, "\sim"] & \mathcal{C}_1 \arrow[d, "q"] \\ * \arrow[r, "j"'] & \mathcal{C}_\infty \end{tikzcd}$$

Theorem:

$$i_\# p^* \xrightarrow{\sim} q^* j_\# .$$

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⑤ **Example:** $\mathfrak{gl}_n$ = $n \times n$ matrices over $\mathcal{C}_1$

$$\begin{tikzcd} & \mathfrak{b} \arrow[dl, "\pi"'] \arrow[dr, "\kappa"] & \\ \mathfrak{h} & & \mathfrak{gl}_n \end{tikzcd}$$

$i_{\#} h \simeq C_\infty h$

Character: $C_\infty h \xrightarrow{C_\infty} S$ $\quad$ $\lambda : h \to S$ map of S-modules.

S_λ

$$\boxed{K_{\#} \pi^{*} S_\lambda = V_\lambda}$$

Verma module.

⑥ <u>Harish-Chandra pairs</u>

$GL_n S$

$GL_n S$ is an <u>S-algebraic group</u>

$$gl_m S^\vee = F(gl_m S, S)$$

$$\det{}^{-1} C_\infty\, gl_m S^\vee = \mathcal{O}_{GL_m S}$$

algebraic group structure:

$$gl_m S^\vee \longrightarrow gl_m S^\vee \wedge gl_m S^\vee$$

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\eta"'] & X \wedge X \arrow[d, "\eta \wedge \eta"] \\
C_\infty X \arrow[r, dashed] & C_\infty X \wedge C_\infty X
\end{tikzcd}$$

GL_n - representation V: E_∞-algebras

$$C_\infty V \longrightarrow C_\infty V \wedge \mathcal{O}_{GL_n S}$$

satisfies coassociativity, co-unit.

To get a Harish-Chandra pair: we need a connection between an S-algebraic group and an S-Lie algebra

Topological Quillen cohomology!

$C_\infty G$ $\qquad\qquad$ $\mathcal{E}_1$ - co-action

$B C_\infty G$ has $\mathcal{E}_2$-co-action

$\Sigma^{-1} C_\infty G$ has $\Sigma^{-1}\mathcal{E}_2$-co-action

Lie

$\vdots$ $\qquad\qquad$ $\shortparallel$

hocolim: Quillen homology : holim $\Sigma^{1-k}\mathcal{E}_k$-coaction

Quillen cohomology : Lie action

⑦ P. Hu rephrased Hussan's "diagram relations" as Verdier duality for Kan extensions

$$\begin{tikzcd}
(\mathfrak{gl}_n,\ GL_{i_1} \times \cdots \times GL_{i_j}) \arrow[d, shift left=2, "\text{forgetful functor}"] \\
(\mathfrak{gl}_n,\ GL_{i_{11}} \times \cdots \times GL_{i_{j\cdots}}) \arrow[u, shift left=2] \arrow[u, bend left=60] \arrow[u, bend right=60]
\end{tikzcd}$$

(derived)

$$i_1 = \sum i_{1l}$$

$$\vdots$$

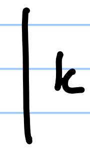
$$i_j = \sum i_{jl}$$

left and right Kan extension on representations

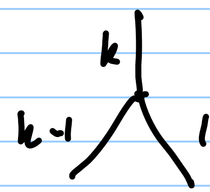
Zuckermann functors (Bernstein)

Calculational questions about dualizing objects.

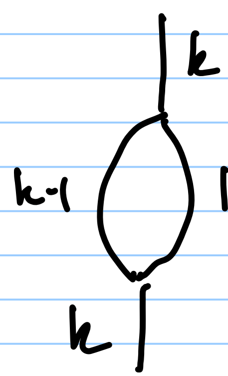
GL_k



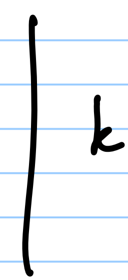
$GL_k \to GL_{k-1} \times GL_1$

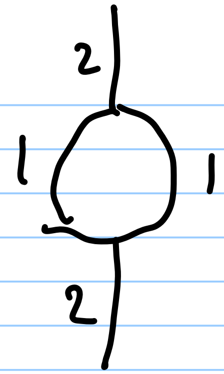


$V_n \quad \varepsilon^?$

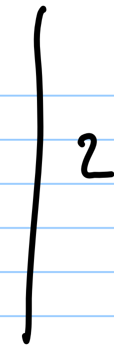


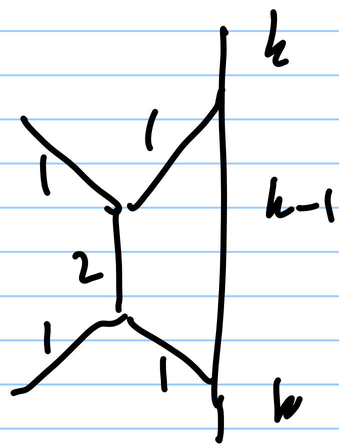
$=$ $[k]$



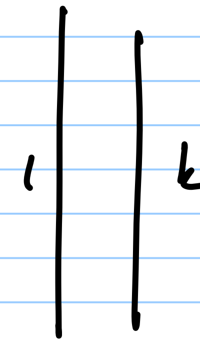


$= [2]$

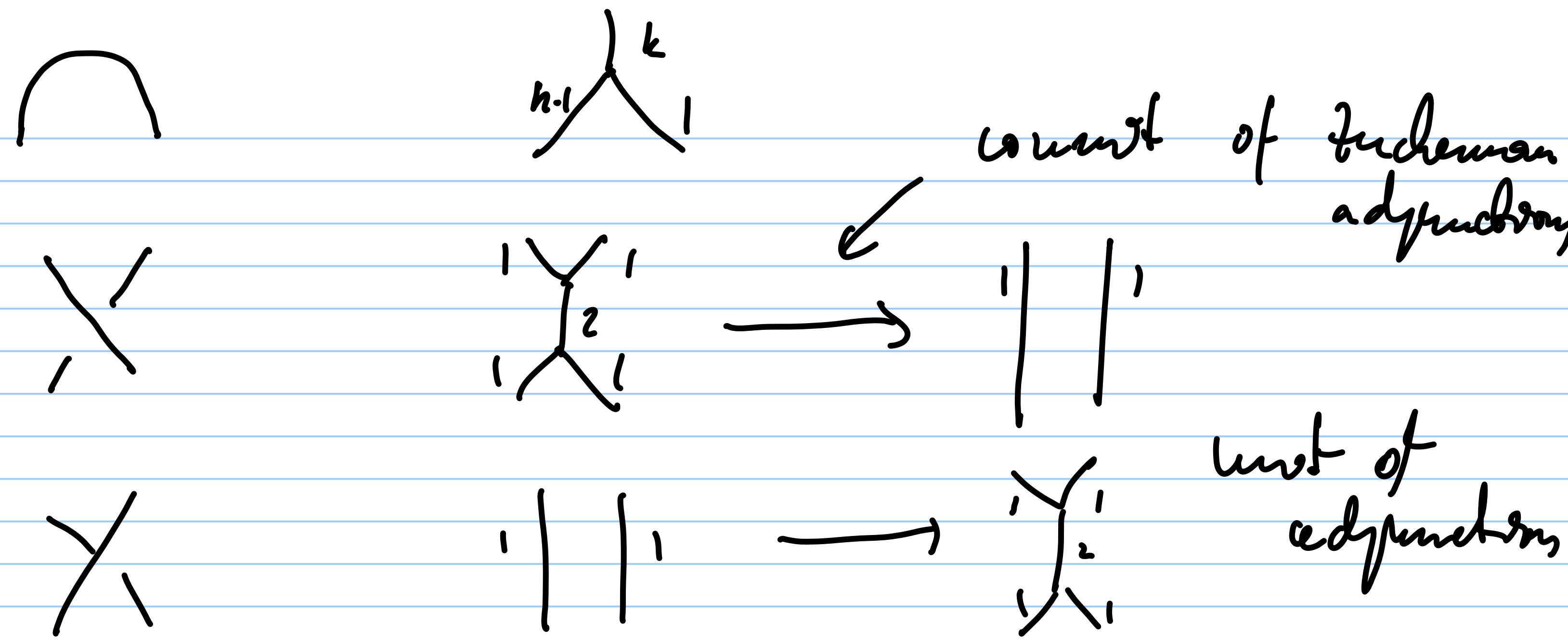




$= [k-1]$



2 more nasty ones.



We can construct an $\mathfrak{sl}_k$ stable homotopy type at a large prime $\gg k$.

8. On characters

$\chi_\infty : C_\infty h \xrightarrow{C_\infty} S$ $\qquad F(h, S)$

$\chi_\infty = \Omega^\infty F(h, S)$

∞ loop space

$\chi = \{ C_\infty h \xrightarrow{C_1} S \}$

$$\Omega \chi = F_{(C_1 C_\infty h)}(S, S) = F_{C_\infty h}(S, F_S(S, S))$$

$$= F_S(S \wedge_{C_\infty h} S, S) =$$

$$= F_S(C_\infty \Sigma S, S)$$

completely calculable
via Carlsson's Thm

$$\mathcal{H} = \underline{\Omega^\infty \Sigma F(C_\infty \Sigma h, S)}$$

Σh

$\Sigma^{-1} F(h, S)$

$F(h, S)$