THE EQUIVARIANT COMPLEX COBORDISM RING OF A FINITE ABELIAN GROUP

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Abstract. We compute the equivariant (stable) complex cobordism ring $(MU_G)_*$ for finite abelian groups G.

1. Introduction

The calculation of the non-equivariant cobordism ring due to Milnor and Quillen [9, 10] was one of the great successes of algebraic topology. The G-equivariant complex cobordism ring for G a compact Lie group can be defined analogously to the non-equivariant case. It was noticed almost immediately however (e.g. [14]) that because of failure of equivariant transversality, equivariant cobordism groups are not the homotopy groups of an $RO(G)$-graded generalized (co)homology theory and hence are extremely difficult to calculate (essentially, suspension spectra arise, so this is comparable in difficulty to, say, the stable homotopy groups of spheres). Because of this, tom Dieck [13] introduced the *stable* equivariant complex cobordism ring), which is the universal object remedying this situation. It has both a geometric characterization (Broöcker and Hook [1]) and a characterization as the coefficient ring of the G-equivariant Thom spectrum.

Perhaps surprisingly, the problem of calculating explicitly tom Dieck's stable equivariant cobordism ring $(MU_G)_*$ has remained open for the last 40 years, despite some great progress (e.g. [3, 4, 5]). To date, there were only two complete calculations known: The case of a p-primary cyclic group was done by the second author [6]. This computation comes in the form of a pullback diagram, but a recipe is given in [6] for recovering explicitly individual elements of the cobordism ring from the diagram. This method was used by Strickland [12] to give, by purely algebraic methods, a presentation of the $\mathbb{Z}/2$-equivariant stable cobordism ring for in terms of commutative ring generators and defining relations.

The first author was supported by an NSF graduate fellowship. The second author was supported by NSF grant DMS 1102614.

The other known computation is due to Dev Sinha [11]. His result is a beautiful presentation of the MU_*-algebra $(MU_{S^1})_*$ in terms of generators and defining relations. This computation, in fact, has the additional benefit that it gives explicit algebra generators of $(MU_{(S^1)^n})_*$, and via a surprising short exact sequence, also generators of $(MU_G)_*$ for any finite abelian group G. Sinha's approach uses Comezana's theorem [2] that $(MU_G)_*$ for a compact Lie group is a *free MU_*-module.* This is used to pick splittings of restriction maps. Comezana's proof is highly non-constructive, and Sinha's generators are therefore, necessarily, non-explicit (from the point of view of [6, 12], it is, for example, not even at all obvious how to write down explicit free generators of $(MU_{\mathbb{Z}/2})_*$ as an MU_*-module). What is remarkable about the main theorem of [11] about $(MU_{S^1})_*$ is that changing generatos within the choices allowed leads to an isomorphism of ring *with relations of the same form.*

The main result of the present note is an explicit calculation of $(MU_G)_*$ for a finite abelian group G. While the meaning of the words "explicit calculation" is debatable in the case of a complicated ring such as $(MU_G)_*$, the answer we give here is purely algebraic, described in terms of concrete ring-theoretic constructions. In fact, the form in which the result appears is a direct generalization of [6], with the pull-back replaced by a more complicated limit diagram. Similar comments as in [6] regarding extracting specific elements apply to the present case, and the method of Strickland [12] can therefore in principle also be applied to our present situation.

To state the result, we must recall certain basic concepts of equivariant homotopy theory ([7]). Recall that a *family* $\mathcal{F}$ of subgroups of a finite group is a system closed under subgroups and conjugation (the latter being vacuous in the abelian case). The *classifying space* of a family $\mathcal{F}$ is a G-CW complex $E\mathcal{F}$ which satisfies

$$E\mathcal{F}^H \simeq \begin{cases} * & \text{if } H \in \mathcal{F} \\ \emptyset & \text{else.} \end{cases}$$

Recall also the homotopy cofiber sequence

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}}$$

where the subscript $+$ means the inclusion of a disjoint base point. We will mostly be interested in two kinds of families associated with a subgroup $H \subseteq G$, namely the family $\mathcal{F}(H)$ of subgroups contained in H and the family $\mathcal{F}[H]$ of subgroups not containing H. Instead of $E\mathcal{F}(H)$, one usually writes EG/H.

Let G be a finite abelian group. Denote by $P(G)$ the poset of all non-empty sets S of subgroups of G which are totally ordered by inclusion:

$$S = \{H_1 \subsetneq H_2 \subsetneq ... \subsetneq H_k\}, \tag{1}$$

with ordering given by inclusion: $S \leq T$ if and only if $S \subseteq T$.

Let X be a G-equivariant spectrum (in this note, we only consider G-equivariant spectra indexed over a complete universe - see [7]). Then define a functor

$$\Gamma = \Gamma_{G,X} : P(G) \to G\text{-spectra}$$

given by

$$\Gamma(S) = F(EG/H_{k+}, \widetilde{E\mathcal{F}[H_k]} \wedge F(EG/H_{k-1+}, ... \wedge F(EG/H_{1+}, \widetilde{E\mathcal{F}[H_1]} \wedge X)...)) \tag{2}$$

where S is as in (1). Note that there is a canonical and natural morphism of G-spectra

$$Y \to F(EG/H_+, \widetilde{E\mathcal{F}[H]} \wedge Y), \tag{3}$$

and the effect of Γ on arrows is defined by iterating these maps. By iterating (3), there is also a canonical natural transformation

$$Const_X \to \Gamma \tag{4}$$

where $Const_X$ is the constant functor on $P(G)$ with value X In the next section, we shall calculate the effect of the functor Γ on coefficients explicitly in the case $X = MU_G$. This is relatively routine, although the statement is technical. Our main result is the following:

Theorem 1. *For $X = MU_G$, applying the coefficient (homotopy groups) functor to (4) induces an isomorphism*

$$MU_{G*} \xrightarrow{\cong} \lim_{\leftarrow} \Gamma(S)_*. \tag{5}$$

It is worth noting that in taking the limit on the right hand side (5), it suffices to take the limit over the restriction of the functor Γ_* to the partially ordered subset $P'(G)$ of $P(G)$ consisting of sets S of cardinality ≤ 2, as this gives the same limit. In fact, an argument based on transitivity of limits shows that we get the same limit when we restrict even further to the subset $P''(G)$ consisting of sets S which have either cardinality 1 or consist of two groups

$$H_1 \subsetneq H_2$$

for which there does not exist a group K which would satisfy

$$H_1 \subsetneq K \subsetneq H_2.$$

Note that since we are dealing with an inverse limit, the validity of the isomorphism in the category of abelian groups automatically implies its validity in the category of commutative rings.

2. Computation of the functor $(\Gamma_{G,MU})_*$

This is essentially a gathering of known facts. First of all, recall that by tom Dieck's result [13, 6], [5], Corollary 10.4, we have

$$(6)\qquad \begin{array}{c}(\widetilde{E\mathcal{F}[H_1]} \wedge MU)_*^{H_1} = \\ MU_*[u_L^{\pm 1}, u_L^{(i)} | i > 0, L \in \overline{H_1^*}]\end{array}$$

where $A^* = Hom(A, S^1)$ and $\overline{A} = A \smallsetminus \{0\}$. For the purposes of this note we don't really nead to know what the classes $u_L^{(i)}$ are, (we set $u_L^{(0)} = u_L$), the only fact we need to know is that under the canonical map of (6) into

$$(\widetilde{E\mathcal{F}[H_1]} \wedge F(EG_+, MU))_*^{H_1} = MU_*[[u_L | L \in H_1^*]]/(u_L +_F u_M = u_{LM}),$$

we have

$$(7)\qquad u_L^{(i)} \mapsto \text{The coefficient of } x^i \text{ in } x +_F u_L$$

(see [6]). Now assuming inductively that we have calculated the coefficients of the H_{j-1}-spectrum

$$(8)\qquad MU_{S,j-1} = (\widetilde{E\mathcal{F}[H_{j-1}]} \wedge F(EG/H_{j-2}, ...\widetilde{E\mathcal{F}[H_1]} \wedge MU)...))^{H_{j-1}},$$

the H_j/H_{j-1}-spectrum (8) is split only if $j = 2$, but in either case the Borel cohomology spectral sequence associated with

$$(9)\qquad F(EG/H_{j-1+}, MU_{S,j-1})_*^{H_j}$$

collapses by evenness, and hence we know (9) has an associated graded object isomorphic to

$$(10)\qquad (MU_{S,j-1})_*^{H_{j-1}} BH_j/H_{j-1}.$$

On the other hand, the coefficient ring is generated by Euler classes, so the precise relations in the ring (9) are not difficult to compute from the formal group law. Furthermore,

$$(11)\qquad (\widetilde{E\mathcal{F}[H_j]} \wedge F(EG/H_{j-1+}, MU_{S,j-1}))_*^{H_j}$$

is obtained from (9) by inverting the Euler classes u_L of irreducible complex representations L of H_j which are non-trivial on H_j.

Explicitly, let R_j, $j = 0, ..., k$ be a set of G/H_j-representatives of the irreducible non-trivial complex H_{j+1}/H_j-representations (we set $H_0 = \{e\}$, $H_{k+1} = G$). Next, consider the ring

$$\begin{aligned}&A_{G,S} = A_S = \\ &MU_*[u_L, u_M^{-1}, u_N^{(i)} | i > 0, \\ &L \in R_0 \amalg ... \amalg R_k, M \in R_0 \amalg ... \amalg R_{k-1}, N \in R_0]\end{aligned}$$

On this ring, define the following topology $\mathcal{T}_{G,S} = \mathcal{T}_S$: a sequence of monomials

$$a_t \prod_{L \in R_1 \amalg ... \amalg R_k} u_L^{n(L,G)} \in A_S$$

with

$$0 \neq a_t \in MU_*[u_L^{\pm 1}, u_L^{(i)} | i > 0, L \in R_0]$$

converges to 0 if and only if there exists a $j = 1, ..., k$ such that

$$n(L,t) \text{ is eventually constant in } t \text{ for } L \in R_i,\ i > j$$

and

$$n(L,t) \xrightarrow[t]{} +\infty$$

for $L \in R_j$. A sequence of elements $p_t \in A_S$ converges to 0 if and only if choosing arbitrary non-zero monomial summands m_t of p_t, the sequence of monomials m_t converges to 0 in t. A set $T \subset A_S$ is closed if and only if the limit of every sequence in T convergent in A_S is in T.

Theorem 2. $\Gamma(S)_*$ *is the quotient of the completion*

$$(A_S)^{\wedge}_{\mathcal{T}_S}$$

by the (closed) ideal $I_S = I_{G,S}$ *generated by the relations*

$$u_{L_1} +_F u_{L_2} = \left(\sum_{i=1}^{m}\right)_F u_{M_i}$$

whenever

$$L_1 L_2 \cong \prod_{i=1}^{m} M_i$$

and there exists a $j = 1, ..., k$ *such that*

$$L_1, L_2 \in R_j,$$

$$M_i \in R_j \amalg ... \amalg R_k.$$

Proof: An induction on $|G|, k$ using the method described in the beginning of this section. For $|G| = 1$ or $k = 1$ the statement is obvious. For a given $k > 1$, first assume $H_k \neq G$.Then filter the ring

$$(A_{G,S})^{\wedge}_{\mathcal{T}_{G,S}}/I_{G,S}$$

by powers of the ideal

$$(u_L | L \in R_k).$$

By definition, the associated graded ring is

$$((A_{H_k,S})^{\wedge}_{\mathcal{T}_{H_k,S}}/I_{H_k,S})[[u_L|L \in R_k]]/(u_L +_F u_M = u_{LM})$$

(with the understanding, of course, that $u_0 = 0$) which, by the induction hypothesis, coincides with (10). The filtrations also coincides with the Borel cohomology spectral sequence, so the statements follows from that spectral sequence. (The Borel cohomology spectral sequence for complex cobordism in the abelian case is quite standard, see e.g. [4].)

When $H_k = G$, we have, by definition,

$$(A_{G,S})^{\wedge}_{\mathcal{T}_{G,S}}/I_{G,S} = (A_{G.S\smallsetminus\{G\}})^{\wedge}_{\mathcal{T}_{G,S\smallsetminus\{G\}}}/I_{G,S\smallsetminus\{G\}}[u_L^{-1}|L \in R_k],$$

which is $\Gamma_G(S)_*$ by the induction hypothesis and (6). $\square$

It remains to compute the effect of Γ on arrows (i.e. inclusions of S), but this is given simply by

$$u_L \mapsto u_L$$

(i.e by these classes being sent to classes of the same name) and by (7), where applicable. Of course, our description of $\Gamma(S)_*$ depended on choices of G/H_j-representatives of irreducible complex H_{j+1}/H_j-representations, so we need to specify how the description changes when we change representatives. For $j > 1$, replacing L by

$$L' = L \prod_{i=1}^{m} M_i$$

with $M_i \in R_{j+1} \amalg ... \amalg R_k$, we may simply use the relation

$$u_{L'} = u_L +_F u_{M_1} +_F ... +_F u_{M_m}.$$

For $j = 1$, we use the relation

$$(u_{L'} +_F x) = u_L +_F (u_{M_1} +_F ... +_F u_{M_m} +_F x)$$

and compare the coefficients at x^i, where the contents of the parenthesis on the right hand side are expanded as a series in x.

3. Proof of the main theorem

First note that the natural transformation (4) gives a canonical morphism of G-spectra

$$\eta_X : X \to \operatorname{holim}_{\leftarrow} \Gamma. \tag{12}$$

We first prove

Theorem 3. *The morphism η_X is an equivalence of G-spectra for any G-spectrum X.*

Proof: An induction on $|G|$. The statement is clearly true for $|G| = 1$, so assume it is true with G replaced by G', $|G'| < |G|$. Denote by $\check{P}(G)$ the partially ordered subset consisting of all sets $S \in P(G)$ such that

$$G \notin S.$$

Denote by $\mathcal{D}$ the diagram

$$\begin{array}{ccc} & & \widetilde{E\mathcal{F}[G]} \wedge X \\ & & \downarrow \\ \operatorname{holim}_{\leftarrow} \Gamma|_{\check{P}(G)} & \longrightarrow & \widetilde{E\mathcal{F}[G]} \wedge \operatorname{holim}_{\leftarrow} \Gamma|_{\check{P}(G)}. \end{array} \tag{13}$$

Then transitivity of homotopy limits gives an equivalence

$$\operatorname{holim}_{\leftarrow} \Gamma \to \operatorname{holim}_{\leftarrow} \mathcal{D}. \tag{14}$$

(Note that $EG/G = *$.) Now for a subgroup $H \subsetneq G$, we have a canonical inclusion $P(H) \subseteq \check{P}(G)$, and if we consider

$$\operatorname{holim}_{\leftarrow} \Gamma|_{P(H)}$$

as a contravariant functor on the poset Q of subgroups $H \subsetneq G$ with respect to inclusion, we have a canonical equivalence

$$\operatorname{holim}_{\leftarrow}(\operatorname{holim}_{\leftarrow} \Gamma|_{P(H)}) \xrightarrow{\sim} \operatorname{holim}_{\leftarrow} \Gamma|_{\check{P}(G)} \tag{15}$$

where the outside homotopy limit on the left hand side of (15) is taken over Q. (This is true with Γ replaced by any functor.) By the induction hypothesis, however, the canonical morphism

$$F(EG/H_+, X) \to \operatorname{holim}_{\leftarrow} \Gamma_{P(H)}$$

is an equivalence for $H \subsetneq G$, so (15) yields a canonical equivalence

$$F(E\mathcal{F}[G]_+, X) = \operatorname{holim}_{\leftarrow} F(EG/?_+, X)|_Q \xrightarrow{\sim} \operatorname{holim}_{\leftarrow} \Gamma|_{\check{P}(G)}. \tag{16}$$

Therefore, if we denote by $\mathcal{E}$ the diagram

$$\begin{array}{ccc} & & \widetilde{E\mathcal{F}[G]} \wedge X \\ & & \downarrow \\ F(E\mathcal{F}[G]_+, X) & \longrightarrow & \widetilde{E\mathcal{F}[G]} \wedge F(E\mathcal{F}[G]_+, X), \end{array} \tag{17}$$

the canonical map

$$\operatorname{holim}_{\leftarrow} \mathcal{E} \to \operatorname{holim}_{\leftarrow} \mathcal{D} \tag{18}$$

is an equivalence, which further obviously commutes with the canonical morphisms from X.

Note, on the other hand, however, that the canonical morphism from X to $\operatorname{holim}_{\leftarrow} \mathcal{E}$ is an equivalence, since $\mathcal{E}$ is the generalized "Tate square" for the family $\mathcal{F}[G]$. (In other words, the fiber of the canonical morphism

$$X \to \widetilde{E\mathcal{F}[G]}$$

maps to the fiber of the bottom row of $\mathcal{E}$ by the canonical equivalence

$$E\mathcal{F}[G]_+ \wedge X \to E\mathcal{F}[G]_+ \wedge F(E\mathcal{F}[G]_+, X),$$

which is an equivalence. □

To prove the "non-derived" statement (5) for $X = MU_G$, we will use induction, which will have to involve a somewhat more general class of spectra. Concretely, by *generalized* MU_G we mean the smallest class of G-equivariant spectra for all G finite abelian which satisfies the following:

(1) MU_G is a generalized MU_G for all G finite abelian.
(2) If R is a generalized MU_G, and $H \subsetneq G$, then

$$\Phi^H R$$

are generalized $MU_{G/H}$ where $\Phi^H(?) = (\widetilde{E\mathcal{F}[H]} \wedge ?)^H$ is the "geometric fixed point functor" (see [7]).
(3) If R is a generalized MU_G, then

$$F(EG_+, R)$$

is a generalized MU_G.

Proposition 4. *The completion theorem* [4]*, and the statements of Section 7 of* [5] *remain valid with* MU_G *replaced by any generalized* MU_G*.*

Proof: Note that generalized MU_G's are formed by starting with MU_Γ for some Γ finite abelian, and then successively applying

$$\Phi^H, \tag{19}$$

or

$$F(EG_+, ?) \tag{20}$$

for certain subquotients H, G of Γ. If only functors of the form (19) are applied in the process, iteration is in fact unnecessary, and we obtain an MU_G-algebra R where R_* is flat over $(MU_G)_*$ by a result of Greenlees ([5], Corollary 10.4). Therefore, the proofs of [4] and [5], Section 7 apply verbatim with MU_G replaced by R.

If, on the other hand, R is a generalized MU_G in whose formation a functor of the form (20) is used at least once, then the coefficients R_* are known by the computation of Section 2 above. In particular, one sees explicitly that Euler classes of representations still generate the augmentation ideal of R_*, and the proofs [4], [5], Section 7, still apply with MU_G replaced by R. $\square$

Proof of Theorem 1: We will prove that the statement of Theorem 1 is valid with MU_G replaced by any generalized MU_G, which we will denote by R. Our proof is by induction on $|G|$. For $|G| = 1$, the statement is obvious. For a given $|G|$, and $\{e\} \neq H \subseteq G$, denote by $\mathcal{M}_H$ the subdiagram of Γ of the form

(21)

$$\begin{array}{ccc} & & ...F(EG/H'_+, \widetilde{E\mathcal{F}[H']} \wedge R) \\ & & \downarrow \\ F(EG/\{e\}_+, \widetilde{E\mathcal{F}[\{e\}]} \wedge R) & \longrightarrow & ...\widetilde{E\mathcal{F}[H']} \wedge F(EG/\{e\}_+, \widetilde{E\mathcal{F}[\{e\}]} \wedge R) \end{array}$$

where H' ranges over all subgroups of G containing H. (Caution: notice the dots. In other words, at the corners we have diagrams indexed over the subset of $P(G)$ containing only sets S in (1) with $H \subseteq H_1$; this is isomorphic to the poset $P(G/H)$. Recall also that, of course, $\widetilde{E\mathcal{F}[\{e\}]} = S^0$.)

Taking homotopy limits of the corners of (21) for a given $H \neq \{e\}$, we obtain the diagram

$$\begin{array}{ccc} & & \widetilde{E\mathcal{F}[H]} \wedge R \\ & & \downarrow \\ F(EG_+, MU) & \longrightarrow & \widetilde{E\mathcal{F}[H]} \wedge F(EG_+, R). \end{array} \tag{22}$$

Taking the union of the diagrams (22) over $H \neq \{e\}$ where we put the canonical arrows between the corresponding upper right and lower right corners induced by inclusions of the subgroups H, is then equivalent to the homotopy limit of the diagram formed by taking the union of the diagrams $\mathcal{M}_H$, which is isomorphic to the diagram Γ. On the other hand, taking homotopy limits over $H \neq \{e\}$ in the upper and lower right corners of (22), we obtain the "ordinary" Tate square for R (as considered for example in [5]):

$$\begin{array}{ccc} & & \widetilde{EG} \wedge R \\ & & \downarrow \\ F(EG_+, R) & \longrightarrow & \widetilde{EG} \wedge F(EG_+, R). \end{array} \tag{23}$$

Now by the induction hypothesis, the coefficients of the upper right and lower right corners of (22) are equal to the inverse limits of the coefficient functor applied to the corresponding parts of the diagram (21). On the other hand, consider the spectral sequences corresponding to the homotopy limits of the upper right and lower right corners (22). By the first sentence of the proof of Lemma 7.2 of Greenlees [5] (which remains valid with MU replaced by R by Proposition 4 above), the vertical arrow of (22) induces an *isomorphism* in filtration degrees ≥ 1 of the E_2-terms of those spectral sequences, and hence these terms may be ignored, and we see that the corners of the (ordinary) Tate diagram for R are obtained as non-derived limits of the corresponding parts of the diagram Γ.

Finally, the homotopy limit of the Tate square can only have a derived term in filtration degree 1, but such a term would create odd degree elements in $(MU_G)_*$, which do not exist by [3, 8]. □

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