Note Title 9/20/2016

Colloquium UM 9/20/2016

Derived Representation Theory and Categorification of sl_k

joint work with Po Hu and Petr Somberg

Homotopy refinement of $\mathbb{Z}$

$$S \longrightarrow \mathbb{Z}$$

Algebraic topology: Generalized (co)homology theory

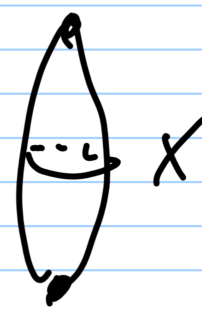
duality

Examples 60's: K-theory (Atiyah) bundles

cobordism theories (Thom)

stable homotopy $\varinjlim \pi_{n+k} \Sigma^k X$

ΣX



$$\tilde{E}^{n+1}(\Sigma X) \cong \tilde{E}^{n}(X)$$

Σ left adjoint functor to Ω

$$\Omega Z = \mathrm{Map}_*(S^1, Z).$$

Representability

based homotopy classes

$$\tilde{E}^n(X) = [X, Z_n]$$

$$Z_n \xrightarrow{\simeq} \Omega Z_{n+1}$$

May spectrum:

(Z_n) $\quad n \in \mathbb{N}_0$

$$Z_n \xrightarrow{\simeq} \Omega Z_{n+1}$$

What is it about a space $Z = Z_0$ that makes it capable to be "delooped" infinitely?

Infinite loop space = abelian group up to homotopy and infinitely many higher homotopies

A homotopy refinement of an abelian group

One machinery which enables this is operads:

$$(\mathscr{C}(n))_{n=0,1,2,\ldots}$$

An algebra over $\mathcal{C}$

$$\mathcal{C}(n) \times X^n \longrightarrow X$$

(spaces)

encodes n-ary operations on X

axioms

$$\mathcal{C}(k) \times \mathcal{C}(n_1) \times \cdots \times \mathcal{C}(n_k)$$
$$\downarrow$$
$$\mathcal{C}(n_1 + \cdots + n_k)$$

Σ_k acts on $\mathcal{C}(k)$

unit $\in \mathcal{C}(1)$

Z is an <u>infinite loop space</u> if Z is an algebra over an E_∞-operad

$\uparrow$

$\mathcal{C}(n) \simeq *.$

$\boxed{\text{defines "abelian group"}}$

$\pi_0(Z)$ is an abelian group.

Similarly, we can define the notion of a commutative ring.

In commutative algebra, we have $\oplus$, $\otimes$.

The analogues in based spaces are

$$X \vee Y = X \amalg Y / *_X \sim *_Y \qquad X \wedge Y = (X \times Y) / (X \times *) \cup (* \times Y)$$

ring structure in cohomology of spaces

$$X \longrightarrow X \wedge X$$

In May spectra, we have a notion of a ring spectrum

(Z_n), $Z_n \xrightarrow{\cong} \Omega Z_{n+1}$

$$E \wedge E \to E$$

not completely canonical

(90's : commutative associative unital operation
(commutative monoid ; rigid commutative ring spectrum)

<u>Example</u> : Stable homotopy theory

$$S = (Z_n)_n \qquad Z_n = \varinjlim \Omega^k S^{n+k}$$

the actual commutative ring up to all higher homotopies :

$$\varinjlim \Omega^k S^k \longrightarrow \mathbb{Z}$$

S

$\pi_0 S$.

?! higher homotopy groups of spheres

What we try to do with commutative (rigid) ring spectra?

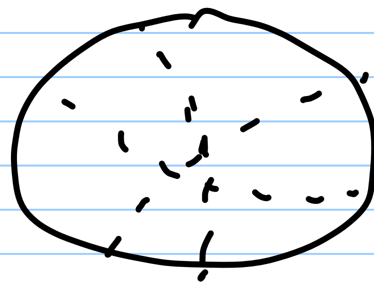
(1) homological algebra - modules, mapping cones, localise etc.

construct new examples

(2) Jacob Lurie : trying to connect it with schemes

$\pi_0 R_i$ ← a scheme

works well when all higher derived functors vanish



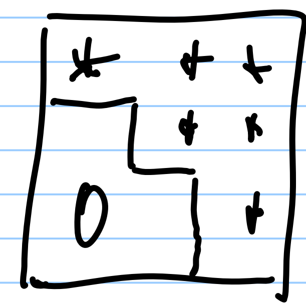
gluing affine schemes from open affine subschemes has no analogue

⇓

<u>topological automorphic forms</u>

③ Representation theory

- Lie algebras $\qquad$ $\mathfrak{gl}_n S$, parabolic subalgebras



work well over

- representations, characters, Verma, coVerma modules, Zuckermann - Bernstein functors, Harish-Chandra theory

Why is this done and how come we can do something?

even over $\mathbb{Z}$, little is known

Motivated by a program in <u>knot theory</u> (categorification)

1985?

Jones polynomial of links $\longrightarrow$ polynomial

$$\langle \emptyset \rangle = 1 \qquad \langle \bigcirc\, L \rangle = (q+q^{-1})\langle L \rangle$$

$$\langle \times \rangle = \langle \asymp \rangle - q\,\langle)(\rangle.$$

(has to do with representations of sl_2).

HOMFLY polynomial $\xrightarrow[\text{similarly related to}]{}$ sl_k

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~ 2000 categorification : Khovanov

Instead of polynomials, q-graded chain cxes.

$$\langle \emptyset \rangle = 0 \to \mathbb{Z} \to 0$$

$$\langle O L \rangle = V \otimes \langle L \rangle$$

$$\langle \times \rangle = \{ 0 \to \langle \asymp \rangle \xrightarrow{d} \langle)(\rangle [1] \to 0 \}$$

Homology = Khovanov homology

Analogue for sl_k (HOMFLY): 2009 Joshua Sussan
(following work of Bernstein,
Frenkel, Khovanov)

described a categorification of representation
theory of sl_k using BGG categories over
gl_n.

representation theory over $\mathbb{C}$.

Newest trend: Stable homotopy categorification

$\sim$ 2012 Lipshitz, Sarkar

sl_2 : Khovanov $\underbrace{\text{stable homotopy type}}_{\text{spectrum}}$

(K., D. Krit, Hu): construction via stable homotopy topological quantum field theories.

The current result is to do it for $\mathfrak{sl}_k$ using the method of Sussan (<u>at a large prime compared to k</u>).

Some of the concepts Sussan uses:

generalized Verma modules: $\mathfrak{gl}_n$-representations

$$\mathfrak{p}^+ = \begin{pmatrix} * & * & * \\ & * & * \\ 0 & & * \end{pmatrix} \qquad \mathfrak{l} = \begin{pmatrix} \boxed{*} & & 0 \\ & \boxed{*} & \\ 0 & & \boxed{*} \end{pmatrix}$$

Levi factor

Generalised Verma module: finite representation W of $\mathfrak{l}$, pull back to $\mathfrak{p}^+$ representation, induce up to $\mathfrak{gl}_n$: $V_{\mathfrak{p},W}$.

<u>The derived $\mathfrak{p}$-BGG category</u>: finite extensions of $V_{\mathfrak{p},w}$.

$\mathfrak{q} \subseteq \mathfrak{p}$ $\qquad$ $\iota : \mathfrak{p}\text{-BGG} \to \mathfrak{q}\text{-BGG}$.

left, right adjoints : left and right Zuckerman functors.

$\nearrow$

I don't know a fully derived construction.

Imitating over S:

All constructions are <u>rigid</u> - then take models and examples, compute