Colloquium, UN

Derived Representation Theory and Categorification of slk
joint work with Po Hu and Petr Somberg

Homotopy refinement of $\mathbb{Z}$

$S \rightarrow \mathbb{Z}$

Algebraic topology: Generalized (co)homology theory duality
Examples 60's: $K$-theory (Atiyah) bundles

cobordism (Thom)

stable homotopy $\lim_{n \to \infty} \pi_{m+n} \Sigma^k X$

$\mathbb{E}^{m+1}(\Sigma X) \cong \mathbb{E}^m(X)$
$\Sigma$ left adjoint functor to $\Omega$

$\Omega \Sigma = \text{Map}_V(S^1, X).$

Representability

$E^n(X) = [X, \mathbb{Z}_m]$

based homotopy class

$\mathbb{Z}_m \xrightarrow{\sim} \Omega \mathbb{Z}_{m+1}$

May spectrum:

$E_n(\mathbb{Z})$  \( n \in \mathbb{N}_0 \)

$\mathbb{Z}_m \xrightarrow{\sim} \Omega \mathbb{Z}_{m+1}$
What is it about a space $\Sigma = \Sigma_0$ that makes it capable to be “delooped” infinitely?

Infinite loop space = abelian group up to homotopy and infinitely many higher homotopies

A homotopy refinement of an abelian group

One machinery which enables this is

$(\mathcal{L}(m))_{m=0,1,2,\ldots}$
An algebra over $\mathcal{E}$

$(\mathcal{E}(\mu) \times \mathcal{E}(\eta) \times \cdots \times \mathcal{E}(\nu))$

spaces

\( \mathcal{E}(\mu, + \cdots + \nu) \)

\( \mathcal{E}_k \) acts on \( \mathcal{E}(k) \)

unit \( \in \mathcal{E}(1) \)

\( \mathbb{Z} \) is an infinite loop space if \( \mathbb{Z} \) is an algebra over an E∞-operad

axioms

encoded many operations on \( \mathcal{E}(\mu) \)}
\[ \pi_0(\mathbb{Z}) \cong \ast. \]

defines "abelian group"

\( \pi_0(\mathbb{Z}) \) is an abelian group.

Similarly, we can refine the notion of a commutative ring.

In commutative algebra, we have \( \oplus \), \( \otimes \). The analogues in based spaces are
In May spectra, we have a notion of a ring spectrum.

\[ \text{Im} \left( \mathbb{Z}_n \right), \mathbb{Z}_n \xrightarrow{\times} \mathbb{S} \mathbb{Z}_{n+1} \]

\[ E \wedge E \rightarrow E \]
and completely canonical in the 90's:

commutative associative
unital operation

(commutative monoid; rigid commutative monoid)

Example: Stable homotopy theory

$S = (S^0)_m, \quad S^m = \lim_{k \to \infty} S^{m+k}$

commutative

the actual way up to all higher homotopies

$\lim_{k \to \infty} S^k \rightarrow \mathbb{Z}$
What we try to do with commutative (rigid) ring spectra?

1. Homological algebra - modules, mapping cones, localization, etc.

Construct new examples
Jacob Lurie: trying to connect it with schemes. \( \Pi_0 R_i \) — a scheme when all higher derived gluing affine functions schemes from open affine subschemes has no analogue.

\[ \text{topological automorphy} \]
(3) Representation theory

- Lie algebras
- $gl_n S$, parabolic subalgebras

\[
\begin{array}{c}
\times \times \\
\times \\
0 \\
\times \\
\times \times \\
\end{array}
\]

- Representations, characters, Verma, coVerma modules, Buchsbaum - Bernstein junctions, Harish-Chandra theory
Why is this done and how come we can do something?  

Motivated by a program in knot theory (categorification)  

1985?  

Jones polynomial of links \[ \mapsto \] polynomial  
\[ \langle \emptyset \rangle = 1, \quad \langle O L \rangle = (q + q^{-1}) \langle L \rangle \]  
\[ \langle X \rangle = \langle X \rangle - q \langle L \rangle. \]
(has to do with representation of $sl_2$).

**HOMFLY polynomial** $\longrightarrow sl_k$

related to

$\sim 2000$ categorification: Khovanov

Instead of polynomials, $q$-graded chain excess.

$\langle \emptyset \rangle = 0 \rightarrow \mathbb{Z} \rightarrow 0$

$\langle 01 \rangle = V \otimes \langle L \rangle$

$\langle \chi \rangle = \langle 0 \rightarrow \langle \omega \rangle \xrightarrow{d} \langle \rangle \{ [1] \rightarrow 0 \} \rangle$
Homology = Khovanov homology

Analogue for $sl_n$ (HOMFLY): 2009 Joshua Sussan
(following work of Bernstein, Frenkel, Khovanov)

described a categorification of representation
theory of $sl_n$ using $B_n$ categories over
$gl_n$. 
Representation theory over $\mathbb{C}$. 

**Newest trend:** Stable homotopy categorification

$\sim 2012$ Lippitz, Sarkar

$\mathfrak{sl}_2$: Khovanov stable homotopy type spectrum

The current result is to do it for $\mathfrak{gl}_k$ using the method of Bussan (at a large norm compared to $k$).

Some of the concepts Bussan uses:

generalized Verma modules: $gl_n$-representations

\[ \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & \star & \star \end{pmatrix} \quad \uparrow \quad \begin{pmatrix} \star & \star & 0 \\ \star & \star & \star \\ 0 & \star & \star \end{pmatrix} \]
Levi factor

generalized Verma module: finite representation \( W \) of \( l \), pull back to \( \mathfrak{p}^* \) representation, induce up to \( \text{gl}_n \) : \( V_{\mathfrak{p}, \mathfrak{w}} \).

The \( \mathfrak{p}^- \text{-BGG} \) category: finite extensions of \( V_{\mathfrak{p}, \mathfrak{w}} \).

\( q \leq p \) \( l : \mathfrak{p}^- \text{-BGG} \rightarrow q^- \text{-BGG} \).

Left, right adjoints: left and right Fahri-Warner functors.
I don't know a fully derived construction.

Imitating one's:

All constructions are rigid - then take molds and examples, compute